Fermions on spontaneously generated spherical extra dimensions

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ABSTRACT: We include fermions to the model proposed in `hep-th/0606021`, and obtain a renormalizable 4-dimensional $SU(N)$ gauge theory which spontaneously generates fuzzy extra dimensions and behaves like Yang-Mills theory on $M^4 \times S^2$. We find a truncated tower of fermionic Kaluza-Klein states transforming under the low-energy gauge group, which is found to be either $SU(n)$, or $SU(n_1) \times SU(n_2) \times U(1)$. The latter case implies a nontrivial $U(1)$ flux on S^2, leading to would-be zero modes for the bifundamental fermions. In the non-chiral case they may pair up to acquire a mass, and the emerging picture is that of mirror fermions. We discuss the possible implementation of a chirality constraint in 6 dimensions, which is nontrivial at the quantum level due to the fuzzy nature of the extra dimensions.

KEYWORDS: Non-Commutative Geometry, Field Theories in Higher Dimensions, Spontaneous Symmetry Breaking.

Contents

1. Introduction

The idea of unification of interactions in higher dimensions is central for many modern developments in the theory of elementary particles and fields, going back to Kaluza-Klein. Recently, a surprising new twist has entered this programme: It was found that extra dimensions can arise effectively within a 4-dimensional renormalizable gauge theory, as an

effective description valid up to some energy scale. This has become known under the name of deconstruction [1].

A strikingly simple realization of the idea of a spontaneous generation of extra dimensions was given in [2], inspired by an earlier work [3]. Since we will extend this model here, we briefly recall the main features of [2]. The model is simply $G = \mathrm{SU}(\mathcal{N})$ Yang-Mills theory on M^4 for some generic (large) $\mathcal{N} \in \mathbb{N}$, with 3 scalars in the adjoint of G transforming as vectors under a global $\mathrm{SO}(3)$ symmetry. It turns out that adding the most general renormalizable potential leads to SSB and to the formation of an extra-dimensional fuzzy sphere via the Higgs effect. The unbroken gauge group is generically $K = \mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1)$, or possibly $K = \mathrm{SU}(n)$. The gauge fields on S_N^2 arise from fluctuations of the $\mathfrak{su}(\mathcal{N})$-valued scalar fields which form the extra-dimensional sphere. For energies less than $\Lambda_{6D} = \frac{N^2}{R}$, the appropriate description of the model is then as Yang-Mills theory with gauge group K on $M^4 \times S_N^2$. Here R is the radius of the internal fuzzy sphere, which is determined (along with the other low-energy parameters including n_1, n_2) by the coupling constants of the model. This interpretation was confirmed by the full harmonic analysis, i.e. by recovering precisely the expected Kaluza-Klein modes, up to the cutoff Λ_{6D}. Above that energy scale, the model again behaves like a 4D gauge theory, thus maintaining renormalizability. The main features of compactification on higher dimensions are hence realized within the framework of renormalizable 4D field theory.

This dynamical or spontaneous generation of extra dimensions is of course strongly suggestive of gravity. Indeed, the results of [4] allow to understand this mechanism in terms of gravity: the scalar potential defines a matrix-model action which - using a slight generalization of [4] - can be interpreted as nonabelian Yang-Mills coupled to dynamical Euclidean gravity in the extra dimensions.

In the present paper, we add fermions to this model, and work out their effective description from both the 6D and 4D point of view. In particular, we show how to obtain a model which has an effective description as Yang-Mills theory on $M^4 \times S_N^2$, with fermions coupling appropriately to the 6D gauge fields and transforming under the unbroken gauge group $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1)$ resp. $\mathrm{SU}(n)$.

In order to make renormalizability manifest, we start again from the 4D point of view, and add fermions transforming appropriately under the symmetries of the bosonic sector. Renormalizability strongly restricts the possible Yukawa couplings between the fermions and the scalar fields. We then determine whether the fermions acquire the expected action for the effective 6-dimensional space $M^4 \times S_N^2$.

We first show in section 3.1 that adding a "minimal" set of fermions does not lead to the desired 6D behavior. However upon doubling the set of fermions, the appropriate 6D picture is indeed found. As shown in detail in section 3.2, the effective description is that of Dirac fermions on $M^4 \times S_N^2$. This is confirmed by explicitly identifying all Kaluza-Klein modes on S_N^2, and determining their masses in the effective 4-dimensional description.

The (generic) case of the low-energy gauge group $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1)$ is particularly interesting. The extra-dimensional sphere then automatically carries a magnetic flux, which couples to the fermions transforming in the bifundamental of $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2)$. According to the index theorem, this implies that these fermions have zero modes, which are expected

to become precisely the massless fermions from the 4D point of view. This conclusion is only true for a chiral 6D theory; in the non-chiral case, two such "would-be zero modes" with opposite chirality can form a massive Dirac fermion.

We study the above mechanism in the present model in section 3.4. The expected (would-be) zero modes are indeed found, in agreement with the theoretical expectations. However since the fermions behave like Dirac fermions on $M^4 \times S_N^2$, these would-be zero modes indeed acquire a mass unless some fine-tuning is imposed. One would therefore like to impose a chirality constraint on the fermions. This is difficult here, because the chirality operator on the fuzzy sphere is a dynamical operator depending on the scalar fields. While a chirality constraint can be imposed on the classical level, its implementation on the quantum level is not clear. Therefore we arrive at a picture of "mirror fermions", where each chiral fermion has a partner with opposite chirality and quantum numbers. Such models have been considered from a phenomenological point of view in [5].

The model shows some intriguing features hinting at a simpler structure at high energies. In particular, we discuss in section 4 an extended $\mathrm{SU}(2\mathcal{N})$ structure which naturally accommodates both the bosonic and the fermionic matter. It also suggests a natural way of obtaining a chiral model, using essentially projector-valued fields. Nevertheless its consistency at the quantum level (i.e. renormalizability) is not clear, and at present it is meant mainly as a stimulation for further research.

There are many possible generalizations and variants of the model discussed here. In particular, we discuss in section 6 a possible mechanism for further symmetry breaking using so-called fluxons on S_N^2, which are non-classical, topologically nontrivial solutions of gauge theory on S_N^2. Generalizations to other fuzzy internal spaces may allow to obtain chiral models. It is also interesting here to recall the analysis of [6], where the spectrum of the standard model has been related to the zero modes of the Dirac operator on other fuzzy spaces; the generation of a nontrivial index on S_N^2 has also been discussed in [7, 8]. Finally, a similar supersymmetric model for spherical deconstruction has already been given in [9]. However, the remarkable mechanism in our model for selecting a single vacuum with particular unbroken gauge group out of the vast number of possibilities is lost there, and a SUSY version preserving this mechanism would be very desirable.

2. The bosonic action

We start by recalling the definition and main features of the model in [2]. Consider the $\mathrm{SU}(\mathcal{N})$ gauge theory on 4-dimensional Minkowski space M^4 with coordinates y^μ, $\mu = 0, 1, 2, 3$, with action

$$\mathcal{S}_{\mathrm{YM}} = \int d^4 y \, Tr \, \left(\frac{1}{4g^2} F_{\mu\nu}^\dagger F_{\mu\nu} + (D_\mu \phi_a)^\dagger D_\mu \phi_a \right) - V(\phi). \tag{2.1}$$

Here A_μ are $\mathfrak{su}(\mathcal{N})$-valued gauge fields, $D_\mu = \partial_\mu + [A_\mu, .]$, and

$$\phi_a = -\phi_a^\dagger, \qquad a = 1, 2, 3 \tag{2.2}$$

are 3 traceless antihermitian scalars in the adjoint of $\mathrm{SU}(\mathcal{N})$,

$$\phi_a \to U^\dagger \phi_a U, \tag{2.3}$$

where $U = \mathrm{U}(y) \in \mathrm{SU}(\mathcal{N})$. Furthermore, the ϕ_a transform as vectors of an additional global SO(3) resp. SU(2) symmetry. $V(\phi)$ is of course the most general renormalizable potential invariant under the above symmetries, which can be written as

$$
\begin{aligned}
V(\phi) = &Tr\, (g_1\phi_a\phi_a\phi_b\phi_b + g_2\phi_a\phi_b\phi_a\phi_b - g_3\varepsilon_{abc}\phi_a\phi_b\phi_c + g_4\phi_a\phi_a) \\
&+\frac{g_5}{\mathcal{N}}\,Tr(\phi_a\phi_a)Tr(\phi_b\phi_b) + \frac{g_6}{\mathcal{N}}Tr(\phi_a\phi_b)Tr(\phi_a\phi_b)
\end{aligned}
\tag{2.4}
$$

$$
= Tr\left(a^2(\phi_a\phi_a + \tilde{b}\,\mathbb{1})^2 + \frac{1}{\tilde{g}^2}\,F_{ab}^\dagger F_{ab}\right) + \frac{h}{\mathcal{N}}\,g_{ab}g_{ab}
\tag{2.5}
$$

for suitable constants $a, b, \tilde{g}, h$, dropping a constant shift. Here

$$
\begin{aligned}
F_{ab} &= [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c = \varepsilon_{abc}F_c, \\
\tilde{b} &= b + \frac{d}{\mathcal{N}}\,Tr(\phi_a\phi_a), \qquad g_{ab} = Tr(\phi_a\phi_b).
\end{aligned}
\tag{2.6}
$$

We also performed a rescaling

$$
\phi'_a = R\,\phi_a, \qquad R = \frac{2g_2}{g_3},
\tag{2.7}
$$

where R has dimension of length; we will usually suppress R and drop the prime. Here $\tilde{b} = \tilde{b}(y)$ is a scalar field, $g_{ab} = g_{ab}(y)$ is a symmetric tensor field under the global SO(3), and $F_{ab} = F_{ab}(y)$ is an $\mathfrak{su}(\mathcal{N})$-valued antisymmetric tensor field which will be interpreted as field strength on the spontaneously generated fuzzy sphere. In this form, $V(\phi)$ looks indeed like the action of Yang-Mills gauge theory on a fuzzy sphere S_N^2 [10–12]. In particular, the term $(\phi_a\phi_a + \tilde{b})^2$ is necessary for the interpretation as a pure YM action on S_N^2 involving only tangential gauge fields, and it determines and stabilizes a unique vacuum.

It is easy to see that at one loop, the parameters $R, a, \tilde{g}, d$ and h are logarithmically divergent, while b and therefore $\tilde{b}$ is quadratically divergent. The gauge coupling g is asymptotically free. A full analysis of the RG flow of these parameters is complicated by the fact that the vacuum and the number of massive resp. massless degrees of freedom depends sensitively on the values of these parameters, with different effective description at different energy scales. This will be discussed next.

2.1 The minimum of the potential and SSB

The mechanism for the generation (or deconstruction) of extra dimensions in this model is based on spontaneous symmetry breaking and the ordinary Higgs effect. We first have to determine the vacuum, i.e. the minimum of $V(\phi)$. This vacuum turns out to have a geometric interpretation as $M^4 \times S_N^2$, breaking SU($\mathcal{N}$) down to a smaller gauge group. The geometric interpretation is confirmed using harmonic analysis, i.e. identification of the Kaluza-Klein (KK) modes. The Higgs effect then induces the appropriate masses of the higher Kaluza-Klein modes of the fuzzy sphere S_N^2.

To determine the minimum of $V(\phi)$ (2.5) turns out to be a rather nontrivial task, and the answer depends crucially on the parameters in the potential. The potential is positive definite provided

$$
a^2 > 0, \qquad \frac{2}{\tilde{g}^2} > 0, \qquad h \geq 0,
\tag{2.8}
$$

which we assume in the following. For suitable values of the parameters in the potential, we can immediately write down the vacuum. Assume $h = 0$ for simplicity. Since $V(\phi) \geq 0$, the global minimum of the potential is then certainly achieved if

$$F_{ab} = [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c = 0, \qquad -\phi_a\phi_a = \tilde{b}, \tag{2.9}$$

because then $V(\phi) = 0$. This implies that ϕ_a is a representation of SU(2), with prescribed Casimir[1] $\tilde{b}$. These equations may or may not have a solution, depending on the value of $\tilde{b}$. Assume first that $\tilde{b}$ coincides with the quadratic Casimir of a finite-dimensional irrep of SU(2),

$$\tilde{b} = C_2(N) = \frac{1}{4}(N^2 - 1) \tag{2.10}$$

for some $N \in \mathbb{N}$. If furthermore the dimension $\mathcal{N}$ of the matrices ϕ_a can be written as

$$\mathcal{N} = Nn, \tag{2.11}$$

then clearly the solution of (2.9) is given by

$$\phi_a = X_a^{(N)} \otimes \mathbb{1}_n \tag{2.12}$$

up to a gauge transformation, where $X_a^{(N)}$ denote the generator of the N-dimensional irrep of SU(2). This can be viewed as a special case of (2.14) below, consisting of n copies of the irrep (N) of SU(2).

For generic $\tilde{b}$, (2.9) cannot be satisfied. The exact vacuum (which certainly exists since the potential is positive definite) can be found by solving the "vacuum equation" $\frac{\delta V}{\delta \phi_a} = 0$,

$$a^2 \left\{ \phi_a, \phi \cdot \phi + \tilde{b} + \frac{d}{\mathcal{N}} Tr(\phi \cdot \phi + \tilde{b}) \right\} + \frac{2h}{\mathcal{N}} g_{ab}\phi_b + \frac{1}{\tilde{g}^2} \left(2[F_{ab}, \phi_b] + F_{bc}\varepsilon_{abc} \right) = 0, \tag{2.13}$$

where $\phi \cdot \phi \equiv \phi_a\phi_a$.

The general solution of (2.13) is not known. However, it is easy to write down a large class of solutions: any decomposition of $\mathcal{N} = n_1N_1 + \ldots + n_hN_h$ into irreps of SU(2) with multiplicities n_i leads to a block-diagonal solution

$$\phi_a = diag\left(\alpha_1 X_a^{(N_1)}, \ldots, \alpha_k X_a^{(N_k)} \right) \tag{2.14}$$

of the vacuum equations (2.13), where α_i are suitable constants which are determined by the equations of motion. We can expect that this Ansatz indeed contains the true vacuum at least for a reasonable range of parameters, because it is known to reproduce all standard "commutative" solutions of YM on S^2 [10, 13]. It turns out that only 2 cases occur:

Type I vacuum. Let N be the dimension of the irrep whose Casimir $C_2(N) \approx \tilde{b}$ is closest to $\tilde{b}$. If furthermore the dimensions match as $\mathcal{N} = Nn$, we expect that the vacuum is given by n copies of the irrep (N), which can be written as

$$\phi_a = \alpha X_a^{(N)} \otimes \mathbb{1}_n. \tag{2.15}$$

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[1]note that $-\phi \cdot \phi = \phi^\dagger \cdot \phi > 0$ since the fields are antihermitian

This is a slight generalization of (2.12), with α being determined through the vacuum equations (2.13). A vacuum of the form (2.15) will be denoted as "type I vacuum". As explained in detail in [2], it should be interpreted as a spontaneously generated extra-dimensional fuzzy sphere S_N^2, where $x_a \sim \frac{1}{N} X_a^{(N)}$ are the coordinates of S_N^2 (8.1). This is confirmed using harmonic analysis, i.e. by decomposing all fields into the correct harmonics on $M^4 \times S_N^2$.

Type II vacuum. In the generic case, the vacuum is expected to consist of several distinct blocks. This necessarily happens if $\mathcal{N}$ is not divisible by the dimension of the irrep whose Casimir is closest to $\tilde{b}$. Assuming that $\tilde{N} = \sqrt{4\tilde{b} + 1}$ is large, it was shown in [2] that the solution with minimal potential among all possible partitions (2.14) is given either by a type I vacuum, or takes the form

$$\phi_a = \begin{pmatrix} \alpha_1 X_a^{(N_1)} \otimes \mathbb{1}_{n_1} & 0 \\ 0 & \alpha_2 X_a^{(N_2)} \otimes \mathbb{1}_{n_2} \end{pmatrix}, \tag{2.16}$$

as long as the integers N_1, N_2 satisfy $\frac{N_i}{\tilde{N}} \approx 1$ and of course $\mathcal{N} = N_1 n_1 + N_2 n_2$. Furthermore, the vacuum turns out to satisfy

$$N_2 = N_1 + 1, \tag{2.17}$$

with uniquely determined N_i and n_i. A vacuum of the form (2.16) will be denoted as "type II vacuum", which is the generic case. Using a rather robust convexity argument [2], one can show that more than 2 different types of blocks N_i do not occur for the vacuum.

6D interpretation. As shown in [2], the fluctuations of the scalars or "covariant coordinates"

$$\phi_a = \alpha X_a + A_a \tag{2.18}$$

together withe the gauge fields A_μ provide the components of a 6D gauge field $A_M = (A_\mu, A_a)$ on $M^4 \times S_N^2$. The effective action from a 6D point of view is that of Yang-Mills on $M^4 \times S_N^2$ with gauge group $SU(n)$ for the type I vacuum, and with gauge group $SU(n_1) \times SU(n_2) \times U(1)$ for the type II vacuum. The latter comes with an induced $U(1)$ magnetic monopole on S_N^2 with monopole number $k = N_1 - N_2$, hence $k = 1$ according to the above analysis. The radial components of the fields ϕ_i on S^2 are very massive due to the term $(\phi_a \phi_a + \tilde{b} \mathbb{1})^2$, and not visible at low energies.

Fluxons. There is a further type of solutions to the equations of motion (2.13), known as fluxon in the context of noncommutative gauge theory. It is given by (one or several) one-dimensional blocks of the form

$$\phi_a = c_a \quad \in i\mathbb{R} \tag{2.19}$$

corresponding to a vector $\vec{c} \in \mathbb{R}^3$. Its length $\vec{c}^2 = \sum_a c_a^2 \approx -\tilde{b}$ is determined by the e.o.m., minimizing the potential. Since the term $(\phi_a \phi_a + \tilde{b} \mathbb{1})^2$ dominates assuming that $a^2 \approx \frac{1}{\tilde{g}^2}$, such a fluxon block contributes typically $S_{\text{fluxon}} \approx \frac{1}{\tilde{g}^2} \vec{c}^2 \approx \frac{1}{\tilde{g}^2} \tilde{N}^2$ to the action through the field strength. This is large compared to the "regular" solutions of type I and type

II vacuum, and was therefore not considered any further in [2]. Nevertheless this may play a role for relatively small $\mathcal{N}$, and the possibility of off-diagonal terms as discussed in section 6 justifies further consideration. There exist further solutions to the equations of motion (2.13), which are however strongly suppressed and not expected to be relevant here.

3. Fermions

We now want to add fermions to our model (2.1). In order to ensure renormalizability we start with the 4-dimensional point of view, and write down the most general renormalizable Lagrangian compatible with the symmetries. The fermion content and their transformation under the symmetries of the above model are chosen such that they have a chance to behave like 6-dimensional fermions in the vacuum corresponding to $M^4 \times S^2$.

Let us briefly summarize the main steps. We start with the minimal case of adding 4D Weyl spinors in the adjoint of SU($\mathcal{N}$) which transform as a doublet of the global SU(2) symmetry. However, it turns out that no 6-dimensional behavior is found, more precisely no kinetic term arises in the extra dimensions.

This problem will be cured by adding a second doublet of 4D Weyl spinors. The most general renormalizable Yukawa interaction then naturally leads to the Dirac operator on the spontaneously generated fuzzy sphere, and the effective description is indeed that of a Dirac fermion on $M^4 \times S^2_N$. In fact, the requirement of renormalizability uniquely singles out the "standard" Dirac operator on S^2_N [14] rather than any of the other candidates that have been proposed in the literature.

The KK-modes and the low-energy properties of the fermions depend of course on the vacuum. In a type I vacuum, the fermions live in the adjoint of the unbroken SU(n) gauge group, and no zero modes are found. In a type II vacuum, the fermions couple to the unbroken SU(n_1) $\times$ SU(n_2) $\times$ U(1) gauge group, and the off-diagonal block components Ψ^{12} and Ψ^{21} then transform in the bifundamental $(n_1) \times (\overline{n}_2)$ resp. $(\overline{n}_1) \times (n_2)$ of SU(n_i) and SU(n_2), with opposite charge under the U(1). Therefore they feel the U(1) magnetic monopole which is induced in that vacuum [10], with opposite charge. The index theorem then applies, and guarantees the existence of "would-be zero modes" for the chiral components of Ψ^{12} and Ψ^{21}. Nevertheless, they may pair up and acquire a mass because the model is non-chiral.

Imposing a chirality constraint corresponding to chiral fermions on $M^4 \times S^2_N$ turns out to be difficult. On the level of the effective action, we discuss 2 possible chirality constraints, which imply the existence of k exact chiral zero modes[2] as expected in a background with magnetic charge k. However, the problem is that the chirality operators on S^2_N necessarily contain the dynamical fields ϕ_a which define the extra dimensions, and this operator has a clear meaning only in or near the geometric vacua. Therefore the implementation of such a chirality constraint on the quantum level is highly nontrivial, and we are not able to define a renormalizable model which describes *chiral* fermions on $M^4 \times S^2_N$. Accordingly we have

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[2]the dynamically preferred vacuum was shown to have $k = 1$ in [2]

a doubling of modes, and the would-be zero modes may pair up to become massive Dirac fermions from the low-energy point of view. This leads to a situation analogous to the "mirror fermions" [5].

The commutative case: fermions on $M^4 \times S^2$. We first recall the classical description of fermions on $M^4 \times S^2$, formulated in a way which will generalize to the fuzzy case. This is done using the embedding $S^2 \hookrightarrow \mathbb{R}^3$ based on the 7-dimensional Clifford algebra

$$\Gamma^A = (\Gamma^\mu, \Gamma^a) = (\mathbb{1} \otimes \gamma^\mu, \sigma^a \otimes i\gamma_5). \tag{3.1}$$

Here σ^a, $a = 1, 2, 3$ generate the 2-resp. 3-dimensional Clifford algebra. The Γ^A act on $\mathbb{C}^2 \otimes \mathbb{C}^4$ and satisfy $(\Gamma^A)^\dagger = \eta^{AB}\Gamma^B$ where $\eta^{AB} = (1, -1, \ldots, -1)$ is the 7-dimensional Minkowski metric. The corresponding 8-component spinors describe Dirac fermions on $M^4 \times S^2$, and can be viewed as Dirac spinors on M^4 tensored with 2-dimensional Dirac spinors on $S^2 \hookrightarrow \mathbb{R}^3$. We can define a 2-dimensional chirality operator χ locally at each point of the unit sphere S^2 by setting

$$\chi = x_a \sigma^a, \tag{3.2}$$

which has eigenvalues ± 1. At the north pole $x_a = (0, 0, 1)$ of S^2 this coincides with $\chi = -i\sigma^1\sigma^2 = \sigma^3$, as expected. This can be understood as usual in terms of a comoving frame, adding a unit vector which is perpendicular to S^2. The action for a Dirac fermion on $M^4 \times S^2$ can then be written as

$$S_{6D} = \int\limits_{M^4} d^4 y \int\limits_{S^2} d\Omega \, \overline{\Psi}_D \left(i\gamma^\mu \partial_\mu + i\gamma_5 \, \slashed{D}_{(2)} + m \right) \Psi_D, \tag{3.3}$$

where

$$\slashed{D}_{(2)} \Psi_D = (\sigma_a L_a + 1)\Psi_D \tag{3.4}$$

is the Dirac operator on S^2 in "global" notation. Here $L_a = i\varepsilon_{abc}x_b\partial_c$ is the angular momentum operator, and the constant 1 in (3.4) ensures $\{\slashed{D}_{(2)}, \chi\} = 0$ and reflects the curvature of S^2. This is equivalent to the standard formulation in terms of a comoving frame, but more appropriate for the fuzzy case.

Chiral (Weyl) spinors $\Psi_\pm$ on $M^4 \times S^2$ are then defined using the 6D chirality operator

$$\Gamma = \gamma_5 \chi, \tag{3.5}$$

and satisfy $\Gamma\Psi_\pm = \pm\Psi_\pm$. They contain both chiralities from the 4D point of view,

$$\Psi_\pm = (0, 1; \pm) + (1, 0; \mp), \tag{3.6}$$

where $(0, 1; \pm)$ denotes a Weyl spinor ψ_α on M^4 with eigenvalue ± 1 of χ, and $(0, 1; \mp)$ a dotted Weyl spinor $\overline{\psi}^{\dot\alpha}$ on M^4 with eigenvalue ∓ 1 of χ. These components are of course mixed under the 6-dimensional rotations.

Majorana spinors on $M^4 \times S^2$ satisfy $\Psi^* = C\Psi$ where C is the 6D charge conjugation operator given by

$$C = i\gamma_2 \sigma_2 \tag{3.7}$$

which satisfies

$$C\Gamma^A C^{-1} = -(\Gamma^A)^*. \tag{3.8}$$

3.1 Minimal Weyl fermions

Consider now our 4-dimensional model (2.1), and let us try to include a doublet of chiral 4-dimensional Weyl spinors

$$\Psi(y) = \begin{pmatrix} \psi_{1,\alpha}(y) \\ \psi_{2,\alpha}(y) \end{pmatrix}, \tag{3.9}$$

which transforms in the fundamental representation of the global SU(2) acting on the index i. Since we want them to behave like spinors on $M^4 \times S_N^2$, the $\psi_{i,\alpha}(y)$ must be $\mathcal{N} \times \mathcal{N}$ matrices; furthermore, since the kinetic term on S_N^2 can arise in our model only through commutators $[\phi_i, .]$ they must transform in the adjoint of SU($\mathcal{N}$). In particular, the anomaly then vanishes. Fermions in the fundamental of SU($\mathcal{N}$) are therefore not considered here.

Thus the $\psi_{1,2}(y)$ are (Grassmann-valued) Weyl spinors which transform under SU($\mathcal{N}$) as $\psi_i(y) \to \mathrm{U}(y)^\dagger \psi_i(y) \, \mathrm{U}(y)$. Then the kinetic term of the action is

$$S_K = \int d^4y \, Tr \, \Psi^\dagger i \gamma^\mu (\partial_\mu + [A_\mu, .]) \Psi = \int d^4y \, Tr \, (\psi_{i,\alpha})^\dagger i (\overline{\sigma}^\mu)^{\dot{\alpha}\beta} (\partial_\mu + [A_\mu, .]) \psi_{i,\beta} \tag{3.10}$$

which is invariant under all the symmetries. It is easy to check that the gauge sector is asymptotically free. Furthermore we should add mass terms and Yukawa couplings, which will lead to Dirac and chirality operators on the fuzzy internal space S_N^2. Renormalizability excludes terms with more than one scalar field ϕ_a. To preserve Lorentz invariance, these term must include 2 unconjugated (or 2 conjugated) spinors. The only possible mass term is

$$S_m = \int d^4y \, Tr \, \psi_{i,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} m \psi_{j,\beta} + h.c. \quad \equiv 0, \tag{3.11}$$

which vanishes due to the Grassmann nature of the spinors (this will no longer true once we double the fermions in section 3.2). However, there exist a non-trivial renormalizable Yukawa interaction

$$S_Y = \int d^4y \, Tr \, \psi_{i,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} (\sigma_a)^{jk} \phi_a \, \psi_{k,\beta} + h.c. \tag{3.12}$$

Using

$$\varepsilon^{ij} (\sigma_a)_j{}^k = \varepsilon^{kj} (\sigma_a)_j{}^i \tag{3.13}$$

it follows that

$$Tr \, \psi_{i,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} (\sigma_a)_j{}^k \phi_a \psi'_{k,\beta} = Tr \, \psi'_{k,\beta} \varepsilon^{\beta\alpha} \varepsilon^{kj} (\sigma_a)_j{}^i \psi_{i,\alpha} \phi_a. \tag{3.14}$$

Therefore (3.12) is in fact the most general renormalizable Yukawa interaction, which can be written as

$$S_Y = \frac{1}{2} \int d^4y \, Tr \, \psi_{i,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} (\sigma_a)_j{}^k \{\phi_a, \psi_{k,\beta}\} + h.c. \tag{3.15}$$

This involve the fuzzy chirality operator (3.20) on S_N^2 as discussed below. On the other hand, the analogous term involving the fuzzy Dirac operator (3.19) on S_N^2 vanishes,

$$Tr \, \psi_{i,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} (\sigma_a)_j{}^k [i\phi_a, \psi_{k,\beta}] \equiv 0. \tag{3.16}$$

Therefore no 6-dimensional behavior is found. This will be cured in the next section.

3.1.1 Dirac operator and chirality on the fuzzy sphere

We collect here the main facts about the "standard" Dirac operator on the fuzzy sphere [14], which is given by the following analog of (3.4)

$$\slashed{D}_{(2)}\Psi = \sigma_a[iX_a, \Psi] + \Psi, \tag{3.17}$$

where $[X_a, X_b] = \varepsilon_{abc}X_c$ generate the fuzzy sphere as explained in appendix 2; recall that X_a is antihermitian here. $\slashed{D}_{(2)}$ acts on 2-component spinors

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3.18}$$

For spinors in the adjoint of the gauge group, the generators X_a are replaced by the covariant coordinates ϕ_a, and the gauged Dirac operator is

$$\begin{aligned} \slashed{D}_{(2)}\Psi &= \sigma_a[i\phi_a, \Psi] + \Psi \\ &= \sigma_a[i\phi_a, \Psi] + \{i\phi_0, \Psi\}. \end{aligned} \tag{3.19}$$

Here we introduce $\phi_0 \equiv -\frac{i}{2}$ for later convenience. This operator will arise automatically in section 3.2, singled out from other possible fuzzy Dirac operators [15–17] by the requirement of renormalizability. For the time being we focus on the simplest case (2.15).

There exists no chirality operator which anticommutes with $\slashed{D}_{(2)}$ and has eigenvalues ± 1; this follows from the spectrum of $\slashed{D}_{(2)}$, which will be determined below (3.33). Nevertheless, there is a clear notion of approximate chirality for a given vacuum at least for the low-lying modes: consider the covariant operator [14]

$$\chi(\Psi) = \frac{1}{N}\,\sigma_a\{i\phi_a, \Psi\} \tag{3.20}$$

$$= \frac{1}{N}\,(\sigma_a\{i\phi_a, \Psi\} + [i\phi_0, \Psi]) \tag{3.21}$$

which is invariant under the global SU(2). To see this, the extended notation

$$\Phi = \phi_a\sigma^a + \phi_0\sigma^0 \tag{3.22}$$

of section 4 is convenient. Then using

$$(N\chi + \slashed{D}_{(2)})\Psi = 2i(\sigma_a\phi_a + \phi_0)\Psi = 2i\Phi\Psi, \tag{3.23}$$

$$(N\chi - \slashed{D}_{(2)})\Psi = 2i(\sigma_a\Psi\phi_a - \Psi\phi_0) \tag{3.24}$$

it follows that

$$2N(\slashed{D}_{(2)}\chi + \chi\slashed{D}_{(2)})\Psi = -4[(\phi_a\phi_a + \phi_0\phi_0), \Psi] - 2i\sigma_a\varepsilon_{abc}\{F_{bc}, \Psi\} \tag{3.25}$$

and thus

$$(\slashed{D}_{(2)}\chi + \chi\slashed{D}_{(2)})\Psi = -\frac{i}{N}\,\sigma_a\varepsilon_{abc}\{F_{bc}, \Psi\} - \frac{2}{N}\,[(\phi_a\phi_a + \phi_0\phi_0), \Psi] \tag{3.26}$$

which is approximately zero, and exactly zero for $F = 0$. Moreover, $\chi^2 \approx -\frac{4}{N^2}\phi_a^2 \propto \mathbb{1}$ using (3.20), at least for low modes. Therefore χ plays the role of a chirality operator on the fuzzy sphere [14]. This can be understood by considering e.g. the north pole ($x_1 \approx x_2 \approx 0, x_3 \approx R$) of S_N^2, where the tangential Clifford algebra is generated by σ_1 and σ_2; then $\chi \approx i\sigma_1\sigma_2 = \sigma_3$. In particular, (3.23) implies that for low modes, χ can be replaced by

$$\chi\Psi \approx \frac{2i}{N}\Phi\Psi. \tag{3.27}$$

The rhs of (3.27) thus provides an interesting alternative definition of chirality on S_N^2, which is related but not identical to the Ginsparg-Wilson approach [17].

The relation (3.26) implies as usual that the eigenvalues $E_{n,\pm}$ come in pairs with opposite sign, except for simultaneous eigenvectors of of χ and $\slashed{D}_{(2)}$ where either χ or $\slashed{D}_{(2)}$ vanish. Indeed, note that $\sigma_a\varepsilon_{abc}\{F_{bc}, \Psi\} \propto \chi\Psi$ for any of the vacua under consideration here; therefore $\chi\Psi$ is an eigenvector of $\slashed{D}_{(2)}$ for any eigenvector Ψ of $\slashed{D}_{(2)}$. This will be worked out explicitly below, and is related to a fuzzy index theorem.

We note the following identities using (3.14)

$$Tr\,\psi_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}(\slashed{D}_{(2)}\psi')_{j,\beta} = -Tr\,\psi'_{k,\beta}\varepsilon^{\beta\alpha}\varepsilon^{ij}(\slashed{D}_{(2)}\psi)_{j,\alpha} = Tr\,(\slashed{D}_{(2)}\psi)_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}\psi'_{j,\beta} \tag{3.28}$$

and

$$Tr\,\psi_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}(\chi\psi)'_{j,\beta} = Tr\,\psi'_{k,\beta}\varepsilon^{\beta\alpha}\varepsilon^{ij}(\chi\psi)_{k,\alpha} = -Tr\,(\chi\psi)_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}\psi'_{j,\beta}. \tag{3.29}$$

Therefore spinor harmonics $\psi_{i,\alpha}$ and $\psi'_{i,\alpha}$ can have a nontrivial pairing only if they have the same eigenvalue of $\slashed{D}_{(2)}$ and the opposite eigenvalue of χ (if applicable). Further, observe that (3.16) amounts to

$$Tr\,\psi_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}(\slashed{D}_{(2)}\psi)_{j,\beta} = 0, \tag{3.30}$$

i.e. the fuzzy Dirac operator drops out, and the Yukawa coupling (3.15) can be written as

$$S_Y = -\frac{iN}{2}\int d^4y\,Tr\,\psi_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}(\chi\psi)_{j,\beta} + h.c. \tag{3.31}$$

Therefore this model does not have the desired 6D limit. This will be corrected below by doubling the fermions, in which case the Yukawa coupling indeed induce the fuzzy Dirac operator. But before doing that, we determine the spectrum of $\slashed{D}_{(2)}$.

3.1.2 The spectrum of $\slashed{D}_{(2)}$ in the type I vacuum

Since $\slashed{D}_{(2)}$ commutes with the SU(2) group of rotations, the eigenmodes of $\slashed{D}_{(2)}$ in the type I vacuum (2.15) are obtained by decomposing the spinors into irreps of SU(2)

$$\begin{aligned}
\Psi \in (2) \otimes (N) \otimes (N) &= (2) \otimes ((1) \oplus (3) \oplus \ldots \oplus (2N-1)) \\
&= ((2) \oplus (4) \oplus \ldots \oplus (2N)) \oplus (\quad (2) \oplus \ldots \oplus (2N-2)) \\
&=: (\quad \Psi_{+,(n)} \quad\quad \oplus \quad\quad \Psi_{-,(n)}).
\end{aligned} \tag{3.32}$$

This defines the spinor harmonics $\Psi_{\pm,(n)}$ which live in the n-dimensional representation of SU(2) denoted by (n) for $n = 2, 4, \ldots, 2N$, excluding $\Psi_{-,(2N)}$. The eigenvalue of $\not{D}_{(2)}$ acting on these states can be determined easily using some SU(2) algebra, see appendix 2:

$$\not{D}_{(2)} \Psi_{\pm,(n)} = E_{\delta=\pm,(n)} \Psi_{\pm,(n)}, \tag{3.33}$$

where

$$E_{\delta=\pm,(n)} \approx \frac{\alpha}{2} \begin{cases} n, \ \delta = 1, & n = 2, 4, \ldots, 2N \\ -n, \ \delta = -1, & n = 2, 4, \ldots, 2N - 2 \end{cases} \tag{3.34}$$

assuming $\alpha \approx 1$; this is exact for $\alpha = 1$. We note that with the exception of $\Psi_{+,(2N)}$, all eigenstates come in pairs $(\Psi_{+,(n)}, \Psi_{-,(n)})$ for $n = 2, 4, \ldots, 2N - 2$, which have opposite eigenvalues $\pm \frac{\alpha}{2} n$ of $\not{D}_{(2)}$. They are interchanged through χ,

$$\chi \begin{pmatrix} \Psi_{+,(n)} \\ \Psi_{-,(n)} \end{pmatrix} = c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_{+,(n)} \\ \Psi_{-,(n)} \end{pmatrix} \tag{3.35}$$

for some $c \approx 1$, by virtue of the anticommutativity relation (3.26). The chirality operator for the top mode vanishes, $\chi(\Psi_{+,(2N)}) = 0$.

3.1.3 The spectrum of $\not{D}_{(2)}$ in a type II vacuum

Consider now a type II vacuum (2.16),

$$\begin{pmatrix} \alpha_1 X_a^{N_1} \otimes \mathbb{1}_{n_1} & 0 \\ 0 & \alpha_2 X_a^{N_2} \otimes \mathbb{1}_{n_2} \end{pmatrix}. \tag{3.36}$$

We decompose the spinors according to this block-structure as

$$\Psi_i = \begin{pmatrix} \Psi_i^{11} & \Psi_i^{12} \\ \Psi_i^{21} & \Psi_i^{22} \end{pmatrix} \tag{3.37}$$

for $i = 1, 2$. The analysis for the diagonal blocks is the same as before, and they describe fermions in the adjoint of SU(n_1) resp. SU(n_2). The off-diagonal blocks however describe fermions in the bifundamental $(n_1) \times (\overline{n}_2)$ of SU(n_1) $\times$ SU(n_2), and those will provide the interesting low-energy sector. For the moment we ignore the extra SU(n_i) structure. Assuming $N_1 \neq N_2$, their decomposition (3.32) into irreps of the global SU(2) now reads

$$\begin{aligned} \Psi_i^{12} \ \in \ (2) &\otimes (N_1) \otimes (N_2) = (2) \otimes ((1 + |N_2 - N_1|) \oplus (3 + |N_2 - N_1|) \oplus \ldots \oplus (N_1 + N_2 - 1)) \\ &= \qquad ((|N_2 - N_1| + 2) \oplus (|N_2 - N_1| + 4) \oplus \ldots \oplus (N_1 + N_2)) \\ &\oplus ((|N_2 - N_1|) \oplus (|N_2 - N_1| + 2) \oplus \ldots \oplus (N_1 + N_2 - 2)) \\ &=: (\Psi_{+,(n)}^{12} \oplus \Psi_{-,(n)}^{12}) \end{aligned} \tag{3.38}$$

defining the spinor harmonics $\Psi_{\pm,(n)}^{12}$ which live in the representation (n) of SU(2). A similar decomposition holds for $\Psi^{21} \in (2) \otimes (N_2) \otimes (N_1)$. Then the spectrum of $\not{D}_{(2)}$ for Ψ^{12} can be worked out as in appendix 2.

For simplicity, we focus here on the (would-be) zero modes, which can be worked out very easily and is the most interesting sector from the low-energy point of view. They are by definition the lowest modes $\Psi^{12}_{-,(k)}$ in the decomposition (3.38) of Ψ^{12}, where $k = |N_1 - N_2|$ corresponds to the magnetic flux induced in this type II vacuum. It follows immediately from $(\not{D}_{(2)}\chi + \chi \not{D}_{(2)})\Psi = O(\frac{1}{N})$ that they are exact or approximate zero modes of $\not{D}_{(2)}$ (up to $O(\frac{1}{N})$ corrections); this can also be checked directly. These are precisely the k zero modes expected from the index theorem in a monopole background with flux k.

The chirality χ for the would-be zero modes can be determined easily. To this end, note that they live in the subspaces

$$\Psi^{12}_{-,(k)} \in (N + k - 1) \otimes (N) \subset ((2) \otimes (N + k)) \otimes (N),$$
$$\Psi^{21}_{-,(k)} \in (N + 1) \otimes (N + k) \subset ((2) \otimes (N)) \otimes (N + k). \qquad (3.39)$$

Note that this involves the "anti-parallel" resp. "parallel" tensor product for the first 2 factors. Therefore $\Phi = \sigma^a \phi_a + \phi_0 \mathbb{1} \approx -\frac{N}{2}$ if acting from the left on $\Psi^{12}_{-,(k)}$, and $\Phi \approx +\frac{N}{2}$ if acting on $\Psi^{12}_{-,(k)}$. Since $\frac{1}{N}\Phi^L$ agrees with χ up to $1/N$, it follows that

$$\chi(\Psi^{12}_{-,(k)}) = c^{12}\,\Psi^{12}_{-,(k)}, \qquad c^{12} \approx -1,$$
$$\chi(\Psi^{21}_{-,(k)}) = c^{21}\,\Psi^{21}_{-,(k)}, \qquad c^{21} \approx 1. \qquad (3.40)$$

The chirality can be computed more generally using (9.5).

3.2 Doubling the fermions

The fact that we did not arrive at the expected 6-dimensional description of fermions in the previous section can be understood as follows: Usually, in order to introduce fermions in 4+2 resp. 4+3 dimensions one starts with the 6- resp. 7-dimensional Clifford algebra

$$\Gamma^A = (\Gamma^\mu, \Gamma^a) = (\mathbb{1} \otimes \gamma^\mu, \sigma^a \otimes i\gamma_5) \qquad (3.41)$$

which act on $\mathbb{C}^2 \otimes \mathbb{C}^4$ and satisfies $(\Gamma^A)^\dagger = \eta^{AB}\Gamma^B$ where $\eta^{AB} = (1, -1, \ldots, -1)$ is the 7-dimensional Minkowski metric. This corresponds to 4-dimensional Dirac fermions tensored with 2-dimensional Dirac fermions. The 6D chirality operator is given by

$$\Gamma = \gamma_5 \chi, \qquad (3.42)$$

where $\chi = -i\sigma_1\sigma_2 = \sigma_3$. In particular, 6-dimensional chiral fermions necessarily contain both chiralities from the 4D point of view. In order to reproduce this in our model, we should therefore start with 4D Dirac fermions, i.e. double the Weyl fermions introduced above. Hence consider a doublet of Weyl fermions[3] $\psi_{i,r;\alpha}(y)$ for $r \in \{1, -1\}$ in the adjoint of $SU(\mathcal{N})$,

$$\Psi = \begin{pmatrix} \psi_{i,1;\alpha} \\ \psi_{i,2;\alpha} \end{pmatrix} \equiv \begin{pmatrix} \rho_{i,\alpha} \\ \eta_{i,\alpha} \end{pmatrix}. \qquad (3.43)$$

---

[3]It is easy to check that the gauge sector remains to be asymptotically free.

They transform as a doublet under $\mathrm{SU}(2)_R$ acting on the $r \in \{1, -1\}$ indices, which may or may not be a symmetry of the action. This $\mathrm{SU}(2)_R$ contains in particular the $\mathrm{U}(1)_R$ symmetry

$$\begin{pmatrix} \rho_{i,\alpha} \\ \eta_{i,\alpha} \end{pmatrix} \rightarrow e^{iaR} \begin{pmatrix} \rho_{i,\alpha} \\ \eta_{i,\alpha} \end{pmatrix} = \begin{pmatrix} e^{ia} \rho_{i,\alpha} \\ e^{-ia} \eta_{i,\alpha} \end{pmatrix} \tag{3.44}$$

which prevents self-couplings. This will later be identified as "vector" $\mathrm{U}(1)$ charge, with generator

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.45}$$

The only non-vanishing mass term is

$$S_m = \int d^4y \, Tr \, m' \, \psi_{i,r,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} \varepsilon^{rs} \psi_{j,s,\beta} + h.c. \tag{3.46}$$

since any symmetric combination vanishes. Here m' might be complex, which will be important below. Note that this term is automatically invariant under the global $\mathrm{SU}(2)_R$.

Now consider the Yukawa couplings. Using (3.14) and the above definitions, the most general Yukawa interaction can be written as

$$\begin{aligned} S_Y = \int d^4y \, Tr \Big(& \psi_{i,r,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} \varepsilon^{rs} \big(e(\slashed{D}_{(2)} - 1)\psi + fR\chi\psi \big)_{k,s,\beta} \\ & + \big(h_1 \rho_{i,\alpha} (\chi\rho)_{j,\beta} + h_2 \eta_{i,\alpha} (\chi\eta)_{j,\beta} \big) \varepsilon^{\alpha\beta} \varepsilon^{ij} \Big) \quad + h.c. \end{aligned} \tag{3.47}$$

for constants e, f, h_i. Imposing the "vector" $\mathrm{U}(1)_R$ invariance implies $h_1 = h_2 = 0$, while imposing the full $\mathrm{SU}(2)_R$ symmetry implies $f = h_i = 0$, leaving the shifted Dirac operator $(\slashed{D}_{(2)} - 1)$ on the internal sphere as only possible Yukawa interaction. We will first consider the $\mathrm{SU}(2)_R$ - symmetric case, and postpone the general case to section 3.5. Redefining the mass parameter $m' = m + e$, we thus obtain[4]

$$S_Y + S_m = \int d^4y \, Tr \, \psi_{i,r,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} \varepsilon^{rs} \big(e \, \slashed{D}_{(2)} \psi + m\psi \big)_{k,s,\beta} \quad + h.c. \tag{3.48}$$

Including ψ_0 in the extended formalism of section 4 naturally suggests that $m = 0$, but we cannot strictly rule out a bare mass term at this point.

Kinetic term The kinetic term of the action is as in (3.10),

$$S_K = \int d^4y \, Tr \, \Psi^\dagger i\sigma^\mu (\partial_\mu + [A_\mu, .]) \Psi = \int d^4y \, Tr \, (\psi_{i,r,\alpha})^\dagger i(\overline{\sigma}^\mu)^{\dot{\alpha}\beta} (\partial_\mu + [A_\mu, .]) \psi_{i,r,\beta} \tag{3.49}$$

which is invariant under all the symmetries. We will now show that the combined action $S_K + S_Y + S_m$ is naturally interpreted as a 6D action for a Dirac fermion on $M^4 \times S^2$, which at low energy behaves like a compactified 4D action on M^4.

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[4]more precisely $m' = m + \frac{e}{R}$; recall that we dropped the radius parameter R

3.2.1 Effective 6D Dirac fermion

We can combine the $r = 1, 2$ components of the 4 Weyl fermions (3.43) into a Dirac fermion,

$$\Psi_D = \begin{pmatrix} \psi_{i,1;\alpha} \\ \varepsilon_{ij}\,\varepsilon^{\dot\alpha\dot\beta}\,(\psi_{j,2;\beta})^\dagger \end{pmatrix} \equiv \begin{pmatrix} \rho_{i,\alpha} \\ \overline{\eta}_i^{\dot\alpha} \end{pmatrix} \equiv \begin{pmatrix} \rho_\alpha \\ \overline{\eta}^{\dot\alpha} \end{pmatrix} \quad \in \mathbb{C}^4 \otimes \mathbb{C}^2 \otimes Mat(\mathcal{N}, \mathbb{C}). \qquad (3.50)$$

Then the kinetic term can be written as

$$S_K = \int d^4y\, Tr\, \overline{\Psi}_D i\gamma^\mu(\partial_\mu + [A_\mu, .])\Psi_D$$

$$= \int d^4y\, Tr \left(\rho_\alpha^\dagger\, i(\overline{\sigma}^\mu)^{\dot\alpha\beta}(\partial_\mu + [A_\mu, .])\rho_\beta + (\overline{\eta}^{\dot\alpha})^\dagger\, i(\sigma^\mu)_{\alpha\dot\beta}(\partial_\mu + [A_\mu, .])\overline{\eta}^{\dot\beta} \right), \; (3.51)$$

where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \qquad (3.52)$$

acts as usual on the Dirac spinors (3.50). The hermitian extensions of the mass term (3.46) with complex mass can be written as

$$S_m = \int d^4y\, Tr\, (\tilde{m} - im')\rho_\alpha^\dagger \overline{\eta}^{\dot\alpha} + (\tilde{m} + im')\overline{\eta}^{\dot\alpha\dagger}\rho_\alpha = \int d^4y\, Tr\, \overline{\Psi}_D(\tilde{m} + i\gamma^5 m')\Psi_D. \; (3.53)$$

Note that $\int d^4y\, Tr\, \overline{\Psi}_D i\gamma^5 \Psi_D$ will become part of the Dirac operator. Furthermore, using (3.28) we have

$$\int d^4y\, Tr\, \psi_{2,j\beta}\varepsilon^{ji}\varepsilon^{\beta\alpha} i\, \slashed{D}_{(2)}\psi_{1,i\alpha} = \int d^4y\, Tr\, (\overline{\eta}^{\dot\alpha})^\dagger i\, \slashed{D}_{(2)}\rho_\alpha$$

$$= - \int d^4y\, Tr\, \psi_{1,j\beta}\varepsilon^{ji}\varepsilon^{\beta\alpha} i\, \slashed{D}_{(2)}\psi_{2,j\alpha}, \qquad (3.54)$$

and the Yukawa couplings (3.47) in the $SU(2)_R$-symmetric case can be written as[5]

$$S_Y = e \int d^4y\, Tr\, \overline{\Psi}_D i\gamma_5(\slashed{D}_{(2)} - 1)\Psi_D = e \int d^4y\, Tr \left(\rho_\alpha^\dagger i(\slashed{D}_{(2)} - 1)\overline{\eta}^{\dot\alpha} - (\overline{\eta}^{\dot\alpha})^\dagger i(\slashed{D}_{(2)} - 1)\rho_\alpha \right). \tag{3.55}$$

The $SU(2)_R$ symmetry is now hidden but still holds due to Grassmann antisymmetry. In particular, note that $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = R$ (3.45) ensures hermiticity, since

$$\left(\int d^4y\, Tr\, \rho_\alpha^\dagger i\, \slashed{D}_{(2)}\overline{\eta}^{\dot\alpha} \right)^\dagger = - \int d^4y\, Tr\, (\overline{\eta}^{\dot\alpha})^\dagger i\, \slashed{D}_{(2)}\rho_\alpha. \qquad (3.56)$$

Writing again $m' = m + e$, we obtain

$$S_K + S_Y + S_m = S_{6D} = \int d^4y\, Tr\, \overline{\Psi}_D \left(i\gamma^\mu(\partial_\mu + [A_\mu, .]) + ei\gamma_5\, \slashed{D}_{(2)} + \tilde{m} + i\gamma_5 m \right) \Psi_D$$

$$\equiv \int d^4y\, Tr\, \overline{\Psi}_D(\slashed{D}_{(6)} + \tilde{m} + i\gamma_5 m)\Psi_D. \qquad (3.57)$$

---

[5] we can assume that e is real by rotating the phases of the fermions if necessary.

Thus apart from the 2 distinct mass parameters, the fermionic action (3.57) can indeed be interpreted as gauged Dirac operator on $M^4 \times S_N^2$. Note again that this is a result, which was not imposed on the model in any way. We remark that the bare Dirac mass is expected to run only weakly, being protected by the (approximate) 6D chiral symmetry.

3.2.2 Fermionic low-energy action and Kaluza-Klein modes

To obtain the appropriate low-energy action in 4 dimension, we should organize the fermions in terms of the eigenmodes (3.33) $\psi_{\pm,(n)}$ of $\not{D}_{(2)}$, i.e. $\not{D}_{(2)}\psi_{\pm,(n)} = E_{n,\pm}\psi_{\pm,(n)}$. Consider first a type I vacuum. Then

$$\int d^4 y\, Tr\, \overline{\Psi}_D i\gamma_5 \not{D}_{(2)} \Psi_D = \int d^4 y\, Tr \sum_{n,\pm} iE_{n,\pm} \left(\rho^{\dagger}_{\pm,(n),\alpha}\overline{\eta}^{\dot{\alpha}}_{\pm,(n)} - (\overline{\eta}^{\dot{\alpha}}_{\pm,(n)})^{\dagger}\rho_{\pm,(n),\alpha} \right) \quad (3.58)$$

using (3.55),

$$(\overline{\eta}^{\dot{\alpha}}_{\pm,(n)})^{\dagger i} = \eta_{\pm,(n);j\beta}\, \varepsilon^{ji}\, \varepsilon^{\beta\alpha},$$

and the orthogonality of the eigenstates, which follows from

$$\int d^4 y\, Tr\, (\not{D}_{(2)}\overline{\eta}^{\dot{\alpha}})^{\dagger}\rho_{\alpha} = \int d^4 y\, Tr\, (\overline{\eta}^{\dot{\alpha}})^{\dagger}\, \not{D}_{(2)}\rho_{\alpha} \quad (3.59)$$

as well as $\int d^4 y\, Tr\, (\chi(\overline{\eta}^{\dot{\alpha}}))^{\dagger}\rho_{\alpha} = \int d^4 y\, Tr\, (\overline{\eta}^{\dot{\alpha}})^{\dagger}\chi(\rho_{\alpha})$. Hence the fermions naturally pair up into 4D Dirac fermions as follows

$$\Psi_{+,D,(n)} = \begin{pmatrix} \rho_{+,(n),\alpha} \\ i\,\overline{\eta}^{\dot{\alpha}}_{+,(n)} \end{pmatrix}, \qquad \Psi_{-,D,(n)} = \begin{pmatrix} \rho_{-,(n),\alpha} \\ i\,\overline{\eta}^{\dot{\alpha}}_{-,(n)} \end{pmatrix}, \quad (3.60)$$

and we obtain

$$S_Y + S_m = e\int d^4 y\, Tr \sum_n \left((E_{n,+} + m)\, \overline{\Psi}_{+,D,(n)}\Psi_{+,D,(n)} + (E_{n,-} + m)\, \overline{\Psi}_{-,D,(n)}\Psi_{-,D,(n)} \right) \quad (3.61)$$

dropping $\tilde{m}$ for simplicity.[6] The sign of $(E_{n,\pm} + m)$ is irrelevant and can be absorbed by a phase rotation of the negative eigenmodes. The kinetic term which can be written as

$$\int d^4 y\, Tr \sum_n \left(\overline{\Psi}_{+,D,(n)}i\gamma^{\mu}(\partial_{\mu} + g[A_{\mu}, .])\Psi_{+,D,(n)} + \overline{\Psi}_{-,D,(n)}i\gamma^{\mu}(\partial_{\mu} + g[A_{\mu}, .])\Psi_{-,D,(n)} \right), \quad (3.62)$$

and we obtain the expected KK tower of massive 4D Dirac fermions with masses

$$m_{\pm,D,n} = e|E_{n,\pm} + m| \neq 0 \quad (3.63)$$

which are non-zero unless m is adjusted very particularly. In particular, there are no massless fermions even if the bare mass $m = 0$, since $E_{n,\pm} \neq 0$ in the type I vacuum. This will change in the type II vacuum, as discussed in section 3.4. In particular note that also the top modes

$$\rho_{+,(2N)}, \quad \eta_{+,(2N)} \quad (3.64)$$

---

[6]$\tilde{m}$ would lead to an additional shift in the KK mass spectrum

form very massive Dirac fermions in 4D in the non-chiral case, and play no role at low energies. Their role in the chiral case is more subtle and discussed below.

Another comment is in order. We recall from [2] that the effective radius of the internal 2-sphere is given by $r_{S^2} = \frac{\alpha}{g} R$, where g is the gauge coupling. According to (3.63) and (3.34), the fermions see the effective radius

$$\tilde{r}_{S^2} = \frac{\alpha}{e} R, \tag{3.65}$$

which differs in general from r_{S^2} and depends on the Yukawa coupling e. This shows that the present framework provides in fact a slight generalization of the conventional compactification, in accord with the 2 mass terms found in (3.57).

3.3 Chirality and Kaluza-Klein modes

We now want to impose a chirality constraint on $M^4 \times S_N^2$. The first attempt might be to impose $\Gamma \Psi = \Psi$ using the 6D chirality operator Γ (3.42). This is however not sensible because $\Gamma^2 \neq 1$. A consistent 6D chirality constraint could be

$$\tilde{\Gamma} \Psi = \Psi, \tag{3.66}$$

where

$$\tilde{\Gamma} = \gamma_5 \tilde{\chi}, \qquad \tilde{\chi} = \chi_+ - \chi_- \tag{3.67}$$

and $\chi_\pm$ denotes the spectral projectors on the positive and negative eigenvectors of χ. Then we can consider chiral 6D fermions which satisfy

$$\tilde{\chi}\rho_\alpha = \rho_\alpha, \qquad \tilde{\chi}\overline{\eta}^{\dot\alpha} = -\overline{\eta}^{\dot\alpha}. \tag{3.68}$$

It is shown in appendix 2 that $\chi(\overline{\eta}^{\dot\alpha}) = -\overline{\chi\eta_\alpha}$, hence this is equivalent to

$$\tilde{\chi}\rho_\alpha = \rho_\alpha, \qquad \tilde{\chi}\eta_\alpha = \eta_\alpha, \tag{3.69}$$

i.e. the same conditions apply to ρ and η. The (would-be) zero modes of the type II vacuum and the top modes have multiplicity one, and are therefore either admitted or dismissed by this chirality constraint.[7] For the other eigenmodes of $\displaystyle{\not{D}}_{(2)}$, the chirality operator χ exchanges the positive and the negative eigenmodes (3.35),

$$\chi\rho_{+,(n),\alpha} = c\rho_{-,(n),\alpha}, \qquad \chi\eta_{+,(n),\alpha} = c\eta_{-,(n),\alpha} \tag{3.70}$$

for some $c \neq 0$, or equivalently

$$\chi\overline{\rho}^{\dot\alpha}_{+,(n)} = -c\overline{\rho}^{\dot\alpha}_{-,(n)}, \qquad \chi\overline{\eta}^{\dot\alpha}_{+,(n)} = -c\overline{\eta}^{\dot\alpha}_{-,(n)}. \tag{3.71}$$

This reduces the degrees of freedom by half, and

$$\Psi_{+,D,(n)} = \begin{pmatrix} \rho_{+,(n),\alpha} \\ i\,\overline{\eta}^{\dot\alpha}_{+,(n)} \end{pmatrix}, \qquad c\Psi_{-,D,(n)} = \begin{pmatrix} \chi\rho_{+,(n),\alpha} \\ i\,\chi\overline{\eta}^{\dot\alpha}_{+,(n)} \end{pmatrix} = \chi\gamma_5\Psi_{+,D,(n)}. \tag{3.72}$$

---

[7]actually the top modes are discarded since χ vanishes

Then the contribution of $\Psi_{-,D,(n)}$ in (3.61) coincides with the one from $\Psi_{+,D,(n)}$, leading to a single multiplet of 4D massive Dirac fermions $\Psi_{+,D,(n)}$. Therefore in the 6D chiral case there is a single multiplet of 4D massive Dirac fermions $\Psi_{+,D,(n)}$ with mass $E_{n,+}$, as opposed to 2 multiplets in the non-chiral case. However, there are no massless fermions in the type I vacuum.

Alternative chirality operator There is an alternative possibility to define a chirality operator on the fuzzy sphere, which is related (but not identical) to the Ginsparg-Wilson approach of [17]. The basic observation is the following: consider the (antihermitian) $2\mathcal{N} \times 2\mathcal{N}$ matrix

$$\Phi = \phi_0 \mathbb{1}_2 + \phi_a \sigma_a \tag{3.73}$$

with $\phi_0 = -\frac{i}{2}$ as in (4.6). It satisfies $\Phi^2 \approx c\,\mathbb{1}$ in and near any of the vacua of interest, and we assume that Φ has no zero eigenvalue. We can thus define

$$\tilde{\Phi} := \frac{i\Phi}{|i\Phi|}. \tag{3.74}$$

Then

$$\tilde{\chi}' \Psi := \tilde{\Phi}\Psi \tag{3.75}$$

is a good chirality operator for the fuzzy sphere, and we can impose the alternative 6D chirality constraint

$$\tilde{\Gamma}'\Psi = \gamma_5 \tilde{\chi}' \Psi = \Psi. \tag{3.76}$$

This is particularly natural if $\Phi^2 \equiv c\,\mathbb{1}$, which is an interesting constraint studied in the SU($2\mathcal{N}$)- extended formalism of section 4. It is easy to see using (3.23) that $\tilde{\chi}'$ agrees with $\tilde{\chi}$ on the low-energy modes. However, $\tilde{\chi}$ leads to a problem with the top modes:

Top modes. Consider the top modes

$$\rho_{+,(2N)}, \quad \eta_{+,(2N)} \tag{3.77}$$

in the chiral case. Their 2D chirality χ vanishes identically,

$$\chi \rho_{+,(2N)} = 0, \tag{3.78}$$

which follows from $(\slashed{D}_{(2)}\chi + \chi\slashed{D}_{(2)})\Psi = O(\frac{1}{N})$ combined with the fact that $\slashed{D}_{(2)} = O(N)$; this can also be computed directly. Therefore the chiral projection $\tilde{\Gamma}\Psi = \Psi$ removes these top modes.

On the other hand, the alternative 6D chirality constraint (3.76) gives $\tilde{\chi}' \rho_{+,(2N)} = \rho_{+,(2N)}$ and the same for η. This can easily be seen by noting that the top modes only contain the maximal spin as seen by the intertwiner Φ. Therefore the constraint $\tilde{\Gamma}'\Psi = \Psi$ preserves $\rho_{+,(2N),\alpha}$ but excludes $\overline{\eta}^{\dot{\alpha}}_{+,(2N)}$ and hence $\eta_{+,(2N),\alpha}$. The surviving $\rho_{-,(2N)}$ cannot acquire any mass and hence form a large massless multiplet; indeed any self-coupling term

$$Tr\rho_{+,(2N),i\alpha}\varepsilon^{ij}\,\varepsilon^{\alpha\beta}\,\rho_{+,(2N),j\beta} \tag{3.79}$$

vanishes identically. Since it does couple to the gauge field, such a large massless multiplet is not acceptable. This is a serious problem with the chirality projector $\tilde{\Gamma}'$, which might be overcome by adding a second "mirror" copy of fermions.

6D Majorana condition. A fuzzy analog of the 6D Majorana condition $\Psi_D^* = C\Psi_D$ amounts in the component form to

$$\rho_{i,\alpha} = \eta_{i,\alpha}. \tag{3.80}$$

This leads to the minimal approach of section 3.1, which did not give the desired 6D interpretation. The reason appears to be again that the fuzzy sphere does see some trace of the embedding 3rd dimension, and therefore does not seem to allow a Majorana condition.

3.4 Type II vacuum: zero modes and chirality

In a vacuum of type (2.16), we decompose the spinors as

$$\Psi_\alpha = \begin{pmatrix} \Psi_\alpha^{11} & \Psi_\alpha^{12} \\ \Psi_\alpha^{21} & \Psi_\alpha^{22} \end{pmatrix} \tag{3.81}$$

according to (2.16). The analysis for the diagonal blocks Ψ^{11}, Ψ^{22} is the same as before. In particular, there are no zero modes even if the bare mass term m vanishes.

The off-diagonal blocks however describe fermions in the bifundamental of $SU(n_1) \times SU(n_2)$, more precisely Ψ^{12} lives in $(n_1) \times (\overline{n}_2)$ of $SU(n_1) \times SU(n_2)$, while Ψ^{21} lives in $(\overline{n}_1) \times (n_2)$. Their arrangement into Dirac fermions is as follows

$$\Psi_D^{12} = \begin{pmatrix} \psi_{i,1;\alpha}^{12} \\ \varepsilon_{ij}\, \varepsilon^{\dot\alpha\dot\beta}\, (\psi_{j,2;\beta}^{21})^\dagger \end{pmatrix} \equiv \begin{pmatrix} \rho_{i,\alpha}^{12} \\ (\overline{\eta}_i^{\dot\alpha})^{12} \end{pmatrix} \equiv \begin{pmatrix} \rho_\alpha^{12} \\ (\overline{\eta}^{\dot\alpha})^{12} \end{pmatrix} \tag{3.82}$$

and similarly for Ψ_D^{21}, noting that the block index 12 resp. 21 gets interchanged by the conjugation. The 4D chirality γ_5 is now manifest. The contribution of these off-diagonal blocks to the Yukawa can be written as

$$S_Y = e \int d^4y\, Tr \sum_{n>0} \left(E_{n,+} (\overline{\Psi}_{+,D,(n)}^{21} \Psi_{+,D,(n)}^{12} + \overline{\Psi}_{+,D,(n)}^{12} \Psi_{+,D,(n)}^{21}) + E_{n,-} (\ldots) \right). \tag{3.83}$$

This is hermitian,

$$\int d^4y\, Tr\, (\overline{\Psi}_D^{21} \Psi_D^{12})^\dagger = \int d^4y\, Tr\, (\Psi_D^{12})^\dagger \gamma_0^\dagger \Psi_D^{21} = \int d^4y\, Tr\, \overline{\Psi}_D^{12} \Psi_D^{21} \tag{3.84}$$

In particular, (3.58) gives

$$\begin{aligned} S_Y &= -e \int d^4y\, Tr \sum_{n,\pm} iE_{n,\pm} \left((\overline{\eta}_{\pm,(n)}^{12;\dot\alpha})^\dagger \rho_{\pm,(n),\alpha}^{21} + (\overline{\eta}_{\pm,(n)}^{21;\dot\alpha})^\dagger \rho_{\pm,(n),\alpha}^{12} \right) + h.c. \\ &= -e \int d^4y\, Tr \sum_{n,\pm} iE_{n,\pm} \left(\eta_{\pm,(n),i,\alpha}^{12} \rho_{\pm,(n),,j\beta}^{21} - \rho_{\pm,(n),i,\alpha}^{12} \eta_{\pm,(n),j,\beta}^{21} \right) \varepsilon^{\alpha\beta} \varepsilon^{ij} + h.c. \end{aligned} \tag{3.85}$$

This makes explicit the $SU(2)_R$ symmetry acting on (ρ, η). Recall that without this doubling, all diagonal KK modes would be massless.

We now want to understand the low-energy sector of the model.

Would-be zero modes: non-chiral case. Let us focus on the (would-be) zero modes $\Psi^{12}_{-,D,(k)}, \Psi^{21}_{-,D,(k)}$ for $k = N_1 - N_2$, which determine the low-energy physics. The $\Psi^{12}_{-,D,(k)}$ provide k families of fermions transforming in $(n_1) \otimes (\overline{n}_2)$ of $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2)$. Similarly, the (would-be) zero modes $\Psi^{21}_{-,D,(k)}$ provides k fermions transforming in $(\overline{n}_1) \otimes (n_2)$ of $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2)$. In the non-chiral case, they are massless only if the bare mass vanishes.

Consider the chirality of these modes. We have seen in (3.40) that they are eigenmodes of the 2D chirality operator with

$$\tilde{\chi}(\Psi^{12}_{-,D,(k)}) = -\Psi^{12}_{-,D,(k)}, \qquad \tilde{\chi}(\Psi^{21}_{-,D,(k)}) = \Psi^{21}_{-,D,(k)}, \tag{3.86}$$

and it follows as usual that they are exact or approximate zero modes of $\slashed{D}_{(2)}$. This is due to the monopole flux with strength k on the fuzzy sphere. We thus get 2 (almost-) massless Dirac fermions $\Psi^{12}_{-,D,(k)}$ and $\Psi^{21}_{-,D,(k)}$. However, despite their special role there is nothing in the non-chiral theory which prevents them from acquiring a mass term of the form $\int Tr \overline{\Psi}^{12}_{-,D,(k)} \Psi^{21}_{-,D,(k)}$. These terms are explicitly present in (3.85), and they vanish only if the bare mass vanishes. While this is natural in the extended $\mathrm{SU}(2\mathcal{N})$ formalism of section 4, it is not forced by any symmetry and therefore amounts to some fine-tuning. This is why we call them "would-be zero modes".

Moreover, even if these would-be zero modes are exactly massless, the low-energy theory is not complex, since every fermion in $(n_1) \otimes (\overline{n}_2)$ has a counterpart in the conjugate representation $(n_2) \otimes (\overline{n}_1)$. We therefore find essentially "mirror fermions", which will be discussed below.

Zero modes: chiral case. Imposing the 6D chirality constraint $\tilde{\Gamma}\Psi_D = -\Psi_D$ implies using (3.69) and (3.40) that $\Psi^{21}_{-,D,(k)}$ i.e. $\rho^{21}_{-,(k),\alpha}$ and $\eta^{21}_{-,(k),\alpha}$ are discarded, since they have the wrong chirality. Then only $\rho^{12}_{-,(k),\alpha}$ and $\eta^{12}_{-,(k),\alpha}$ survive which both live in $(n_1) \times (\overline{n}_2)$ (or equivalently $\overline{\rho}^{\dot\alpha 21}_{-,(k)}$ and $\overline{\eta}^{\dot\alpha 21}_{-,(k)}$ which live in $(n_2) \times (\overline{n}_1)$), and there is no way to write down a mass term for these modes. Hence we have a doublet of *exactly massless chiral fermions*. This is the desired mechanism based on the index theorem. The reason for the doubling encountered here is that we started with Dirac fermions in the adjoint of $\mathrm{SU}(\mathcal{N})$.

In the case of unbroken gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2)$, these zero modes could be interpreted as k left-handed quarks $(u,d)_L$. The $\mathrm{U}(1)$ generator is $\propto (\frac{1}{3N_1}, \frac{1}{3N_1}, \frac{1}{3N_1}, -\frac{1}{2N_2}, -\frac{1}{2N_2})$, which for large $N_1 \approx N_2$ is close to but different from the standard unbroken $\mathrm{U}(1)$ as obtained e.g. from the $\mathrm{SU}(5)$ GUT. By extending[8] the basic gauge group to $\mathrm{U}(\mathcal{N})$, one could obtain the more standard $\mathrm{U}(1)$ generator $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{2})$. Further prospects to obtain realistic models are discussed in section 6.

Top modes. Consider now the top modes

$$\rho_{-,(N_1+N_2)}, \quad \eta_{-,(N_1+N_2)} \tag{3.87}$$

for the type II vacuum. In the non-chiral case, these top modes form very massive Dirac fermions in 4D which play no role at low energies.

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[8]this is particularly natural from the point of view of noncommutative field theory

In the chiral case, the same remarks as for the type I vacuum apply to the diagonal blocks, leading to large massless multiplets in the case of the $\tilde{\Gamma}'$ projector. The off-diagonal top modes could acquire a mass even using the $\tilde{\Gamma}'$ projector, since $\rho^{12}_{-,(N_1+N_2),i\alpha}$ and $\rho^{21}_{-,(N_1+N_2),i\alpha}$ have the same $\tilde{\Gamma}$ -chirality, and

$$Tr\, \rho^{12}_{-,(N_1+N_2),i\alpha}\varepsilon^{ij}\, \varepsilon^{\alpha\beta}\, \rho^{21}_{-,(N_1+N_2),j\beta} \neq 0 \tag{3.88}$$

is non-vanishing. Therefore only the diagonal ones remain to be problematic. For the $\tilde{\Gamma}$ -chirality, all top modes are projected out.

3.5 Breaking $SU(2)_R$

One of the motivations for the global $SU(2)_R$ symmetry is the fact that the only possible mass term (3.46) is automatically invariant under $SU(2)_R$. However, the Yukawa couplings (3.47) do not necessarily preserve it. Let us now consider the case where the Yukawa couplings break $SU(2)_R$. This is interesting in particular because then the doubling of the fermions becomes asymmetrical, i.e. the "mirror" fermions may have different low-energy properties. This is of course essential from a phenomenological point of view.

The most general Yukawa coupling was given in (3.47). Using

$$Tr\, \rho_{i,\alpha}\chi(\eta_{j,\beta})\varepsilon^{\alpha\beta}\varepsilon^{ij} = Tr\eta_{i,\alpha}\chi(\rho_{j,\beta})\varepsilon^{\alpha\beta}\varepsilon^{ij},$$
$$Tr\, \rho_{i,\alpha}\eta_{j,\beta}\varepsilon^{\alpha\beta}\varepsilon^{ij} = -Tr\eta_{i,\alpha}\rho_{j,\beta}\varepsilon^{\alpha\beta}\varepsilon^{ij} \tag{3.89}$$

it simplifies as

$$S_Y = \int d^4y\, Tr\, \psi_{i,r,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}\varepsilon^{rs}(\slashed{D}_{(2)} - 1)\psi_{k,s,\beta}$$
$$+\varepsilon^{\alpha\beta}\varepsilon^{ij}\Big(2f\rho_{i,\alpha}\chi(\eta_{j,\beta}) + h_1\rho_{i,\alpha}\chi(\rho_{j,\beta}) + h_2\eta_{i,\alpha}\chi(\eta_{j,\beta})\Big). \tag{3.90}$$

Together with the mass term, this gives the general mass matrix for the (would-be) zero modes of $\slashed{D}_{(2)}$ in a type II vacuum

$$\begin{pmatrix} h_1 & m + f\chi \\ m - f\chi & h_2 \end{pmatrix} \tag{3.91}$$

acting on $\begin{pmatrix} \rho \\ \eta \end{pmatrix}$. For illustration, consider the case $h_1 = h_2 = 0$. One can then write the additional term as 6D pseudo-scalar

$$\int d^4y\, Tr f\, \overline{\Psi}\, i\Gamma\, \Psi. \tag{3.92}$$

For the would-be zero-modes in a type II vacuum, this gives explicitly the additional terms

$$\int d^4y\, Tr\, \Big((m + f)\, \rho^{12}_{i,\alpha}\eta^{21}_{j,\beta} + (m - f)\, \rho^{21}_{i,\alpha}\eta^{12}_{j,\beta}\Big)\varepsilon^{\alpha\beta}\varepsilon^{ij} \tag{3.93}$$

If e.g. $m = \pm f$, then (ρ^{12}, η^{21}) form a massive Dirac fermion, while (ρ^{21}, η^{12}) remain massless and form a single mirror pair of fermions with opposite chirality. Therefore this removes[9] the fermion doubling, but it does not remove the fact that each fermion has a mirror partner with opposite chirality and opposite quantum numbers.

4. Extended model and SU($2\mathcal{N}$) structure

In this section we point out that the degrees of freedom of this model are naturally arranged in SU($2\mathcal{N}$) representations. For the fermions this is elaborated in section (4.1). This gives a natural relation with the twisted picture discussed in [9]. Similarly, the scalars ϕ_i are naturally arranged as $\Phi = \phi_0 + \phi_a \sigma_a$ by adding a further component ϕ_0; this has been anticipated several times. There are 2 motivations for this point of view: first, it naturally leads to the correct constant shift in the fuzzy Dirac operator (3.17), see (4.7) below; however this is only suggestive. The main motivation is that it suggests a SU($2\mathcal{N}$)-invariant constraint $\Phi^2 \propto \mathbb{1}$, which is known to provide an alternative description of Yang-Mills theory on S_N^2 [13]. This in turn is related to the alternative definition of a chirality projection (3.76), (5.1). While we are not able at present to show that it is consistent at the quantum level, this appears to be the best candidate for a chirality constraint. All this points to an underlying SU($2\mathcal{N}$) structure, which certainly deserves further investigation, including possible SUSY versions.

4.1 SU($2\mathcal{N}$) structure and Yukawa coupling

It is natural to collect the fermions into a $2\mathcal{N} \times 2\mathcal{N}$ matrix as follows

$$\Psi_\alpha = \psi_{ir,\alpha} = \begin{pmatrix} \psi_{i=1,r=1} & \psi_{i=1,r=2} \\ \psi_{i=2,r=1} & \psi_{i=2,r=2} \end{pmatrix}_\alpha = \begin{pmatrix} \psi_{i,r=1} \\ \psi_{i,r=2} \end{pmatrix}_\alpha \tag{4.1}$$

cf. (3.50), which under the global SU(2) $\times$ SU(2)$_R$ transforms as

$$\Psi \to U \Psi U_R^T. \tag{4.2}$$

Now define

$$\tilde{\Psi} = \sigma_2 \Psi \sigma_2, \qquad \tilde{\psi}^{ir} = \varepsilon^{ij} \psi_{js} \varepsilon^{rs}, \tag{4.3}$$

which using $U^T \sigma_2 U = \sigma_2$ transforms as

$$\tilde{\Psi} \to \sigma_2 U \tilde{\Psi} U_R^T \sigma_2 = U^{-1T} \sigma_2 \tilde{\Psi} \sigma_2 U_R^{-1}. \tag{4.4}$$

Note that σ_2 is the charge conjugation matrix for SU(2). Then the mass term can be written as

$$S_m = \int d^4 y \, Tr \, \psi_{i,r,\alpha} \varepsilon^{\alpha\beta} \varepsilon^{ij} \varepsilon^{rs} \psi_{j,s,\beta} = \int d^4 y \, Tr \, \tilde{\Psi}_\alpha^T \varepsilon^{\alpha\beta} \Psi_\beta \tag{4.5}$$

(+ h.c.). This suggests to arrange the scalars similarly: consider the antihermitian $2\mathcal{N} \times 2\mathcal{N}$ matrix

$$\Phi = \phi_0 \sigma_0 + \phi_a \sigma_a = \phi_\mu \sigma_\mu \tag{4.6}$$

including an additional component ϕ_0, which we set $\phi_0 \equiv -\frac{i}{2}$ for now but which will be allowed to be dynamical later. The Yukawa coupling looks much nicer in this extended formalism:

$$S_Y = 2i \int d^4 y \, Tr \, \tilde{\Psi}_\alpha^T \Phi \, \Psi_\beta \varepsilon^{\alpha\beta} = \int d^4 y \, Tr \, \tilde{\Psi}_\alpha^T (\sigma_a \, [i\phi_a, \Psi_\beta] + \{i\phi_0, \Psi_\beta\}) \varepsilon^{\alpha\beta}$$
$$= \int d^4 y \, Tr \, \tilde{\Psi}_\alpha^T \, \slashed{D}_{(2)} \, \Psi_\beta \varepsilon^{\alpha\beta}, \qquad (4.7)$$

where $\slashed{D}_{(2)}$ is precisely the Dirac operator on the fuzzy sphere, with the correct constant shift due to $\phi_0 = -\frac{i}{2}$ (which will be understood naturally below). We can also impose the discrete symmetry

$$\psi \to i\psi, \qquad \Phi \to -\Phi, \qquad (4.8)$$

which excludes the mass term (4.5), and requires the potential for ϕ_μ below to be even. This is very appealing, since the correct constant shift for the Dirac operator on S_N^2 is then automatic, and no bare mass term is allowed. It strongly suggests an underlying $SU(2\mathcal{N})$ structure. However, the Yukawa coupling (4.7) explicitly breaks $SU(2\mathcal{N})$ down to $SU(\mathcal{N}) \times SU(2) \times SU(2)_R$. Then there is another Yukawa coupling compatible with this unbroken symmetry,

$$\int d^4 y \, Tr \, \tilde{\Psi}_\alpha^T \phi_0 \, \Psi_\beta \varepsilon^{\alpha\beta}. \qquad (4.9)$$

This essentially amounts again to a mass term in our vacua, spoiling to some extent the reason for introducing the $SU(2\mathcal{N})$ structure. One could argue that this term is absent at a fundamental very high scale, and will be induced only at lower scales due to renormalization, where this $SU(2\mathcal{N})$ symmetry is broken.

Further interesting possibilities appear once we include another fermion κ with the same properties as Ψ as discussed in section 5. This might then allow to break $SU(2\mathcal{N})$ spontaneously. However, we leave such explorations to future work.

Relation with twisted picture. This extended formalism suggests to consider the diagonal "twisted" $SU(2)_D \subset SU(2) \times SU(2)_R$ subgroup generated by (U, U^{-1T}). Decomposing Ψ into vector and scalar fermions $\Psi = \Psi_V + \Psi_S$ under this $SU(2)_D$, where

$$\Psi_{V,\alpha} = \psi_{a,\alpha} \sigma_a, \qquad \Psi_{S,\alpha} = \psi_{0,\alpha} \sigma_0, \qquad (4.10)$$

the Yukawa coupling (4.7) can be rewritten as[10]

$$S_Y = 2i \int d^4 y \, Tr \, \tilde{\Psi}_\alpha^T \Phi \, \Psi_\beta \varepsilon^{\alpha\beta}$$
$$= i \int Tr \, (-\Psi_{V,\alpha} \{\Phi, \Psi_{V,\beta}\} - 2\Psi_{S,\alpha} \, [\Phi, \Psi_{V,\beta}] + \Psi_{S,\alpha} \{\Phi, \Psi_{S,\beta}\}) \varepsilon^{\alpha\beta} \qquad (4.11)$$

Now

$$i \int Tr \, \Psi_{V,\alpha} \{\Phi, \Psi_{V,\beta}\} \varepsilon^{\alpha\beta} = 2 \int d^4 y \, Tr \, \psi_{a,\alpha} \, \tilde{\slashed{D}}_{(2)} \psi_{a,\beta} \, \varepsilon^{\alpha\beta}, \qquad (4.12)$$

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[10]Note that the transposition acts only on the 2×2 block structure.

where

$$\left(\tilde{\not{D}}_{(2)}\psi\right)_{a,\alpha} = -\varepsilon_{abc}\left[\phi_b,\psi_{c,\alpha}\right] + \psi_{a,\alpha} \tag{4.13}$$

is the "vector-Dirac operator". Similarly,

$$i\int d^4y\, Tr\, \Psi_{S,\alpha}\left[\Phi,\Psi_{V,\beta}\right]\varepsilon^{\alpha\beta} = 2i\int d^4y\, Tr\,\psi_{0,\alpha}\left[\phi_a,\psi_{a,\beta}\right]\varepsilon^{\alpha\beta} \tag{4.14}$$

and

$$i\int d^4y\, Tr\, \Psi_{S,\alpha}\{\Phi,\Psi_{S,\beta}\}\varepsilon^{\alpha\beta} = \int d^4y\, Tr\,\psi_{0,\alpha}\psi_{0,\beta}\varepsilon^{\alpha\beta} \tag{4.15}$$

The kinetic term of the action is

$$\begin{aligned}
S_K &= \int d^4y\, Tr\, \Psi^\dagger i\sigma^\mu(\partial_\mu + [A_\mu,.])\Psi \\
&= \int d^4y\, Tr\, (\psi_{a,\alpha})^\dagger i(\sigma^\mu)_{\dot\alpha}{}^\beta(\partial_\mu + [A_\mu,.])\psi_{a,\beta} + (\psi_{0,\alpha})^\dagger i(\sigma^\mu)_{\dot\alpha}{}^\beta(\partial_\mu + [A_\mu,.])\psi_{0,\beta}.
\end{aligned}$$

The 6-dimensional interpretation of this form is less obvious, because it looks like a vector on the internal sphere rather than a spinor. As discussed in [9], it can be interpreted as a twisted compactification [18, 19], which is realized very naturally here. The decomposition into KK modes and the low-energy effective action can be computed easily, by decomposing these adjoint spinors into the eigenmodes of the "vector-Dirac operator". We see that this is simply a different organization of our doubled fermion picture.

A truly different model would be obtained if only the twisted $SU(2)_D$ is a symmetry while the full $SU(2) \times SU(2)_R$ is broken. This would be consistent with additional constraints such as $\Psi_0 = 0$, $[\Phi_a,\Psi_a] = 0$, and

$$\tilde\Psi = -\Psi^T. \tag{4.16}$$

However, this still does not give a chiral theory, since these constraints are real.

4.2 $SU(2\mathcal{N})$-extended scalar sector

The above $SU(2\mathcal{N})$ formalism is very interesting for several reasons, and we discuss here how the scalar sector can be extended to be invariant under this $SU(2\mathcal{N})$. Following [10, 13], we consider the antihermitian $2\mathcal{N} \times 2\mathcal{N}$ matrix

$$\Phi = \phi_0\sigma_0 + \phi_a\sigma_a \tag{4.17}$$

as in (4.6), including a scalar field $\phi_0 = -\phi_0^\dagger$ in the adjoint of $SU(\mathcal{N})$. For the moment we assume $\phi_0 = -\frac{i}{2}$, but ϕ_0 will become dynamical below. The main observation is

$$\Phi^2 = (\phi_a\phi_a + \phi_0\phi_0)\mathbb{1} + \frac{1}{2}i\varepsilon_{abc}F_{bc}\sigma^a \tag{4.18}$$

with the generalized S_N^2 field strength

$$F_{ab} = [\Phi_a,\Phi_b] - i\varepsilon_{abc}\{\phi_0,\phi_c\}. \tag{4.19}$$

We note in particular that any vacuum of the form (2.14), we have

$$\Phi^2 = (\phi_a \phi_a + \phi_0^2) \mathbb{1}_2 \propto \mathbb{1}_{2\mathcal{N}} \tag{4.20}$$

for a suitable $\phi_0 \approx -\frac{i}{2}$. We can then furthermore impose the discrete symmetry (4.8), which requires the potential to be even and excludes a mass term for the fermions.

We now promote ϕ_0 to a dynamical field, and consider the extended $SU(2\mathcal{N})$ symmetry

$$\Phi \to U^{-1}\Phi U, \qquad U \in SU(2\mathcal{N}). \tag{4.21}$$

The most general potential compatible with this symmetry is given by

$$V(\Phi) = Tr(M(\Phi\Phi + b^2\mathbb{1})^2) + \text{(double-trace terms)}, \tag{4.22}$$

where the double-trace terms[11] have the form $c_1 Tr(\Phi^2)Tr(\Phi^2) + c_2 Tr(\Phi)Tr(\Phi^3) + c_3(Tr(\Phi))^2 + c_4(Tr(\Phi))^4 + c_5(Tr(\Phi))^2 Tr(\Phi^2) + c_6(Tr(\Phi))^2$. The corresponding equation of motion has the form $a\Phi^3 + b\Phi^2 + c\Phi + d = 0$, where the coefficients may involve trace terms. We assume furthermore that the term $M(\Phi\Phi + b^2\mathbb{1})^2)$ dominates i.e. $M \to \infty$, while the double-trace terms are naturally suppressed by $\frac{1}{\mathcal{N}}$. Then the vacuum Φ decomposes into blocks which are small deformations of

$$\Phi^2 \propto \mathbb{1}, \tag{4.23}$$

which as shown in [13] is naturally interpreted as fuzzy sphere with some gauge group $U(n)$. All these solutions are degenerate with $V(\Phi) = 0$ as long as $V(\Phi) = Tr(a(\Phi\Phi + b\mathbb{1})^2)$. However, due to the presence of e.g. Yukawa terms which break the $SU(2\mathcal{N})$ symmetry, we expect after renormalization additional terms to be induced in the potential for ϕ_α such as those appearing in (2.4). These as well as the double-trace terms of (4.22) give different energy to different solutions with different block structure, completely analogous to the mechanism discussed in section 2.1. Again, it is very plausible that the same convexity argument generically leads to the same types of vacua with low-energy $SU(n_1) \times SU(n_2) \times U(1)$ gauge symmetry.

4.2.1 Constrained scalars

In order to implement the chirality operator $\tilde{\chi}'$, it would be nice to impose a constraint of the form (4.23), which together with a suitable trace condition on $Tr(\Phi)$ provides precisely the tangential gauge fields on the fuzzy sphere [13]. A natural way to impose such a constraint is by adding the following renormalizable term to the action

$$S = TrM(\Phi^2 + c_\Lambda^2)^2 \tag{4.24}$$

and letting $M \to \infty$. This will indeed impose the desired constraint $\Phi^2 = -c_\Lambda^2 \mathbb{1}$ for $M \to \infty$, but we see that a running of c_Λ must be allowed. Of course renormalization

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[11]some of those would be eliminated by fixing the trace of Φ, which however is not compatible with the discrete symmetry (4.8).

will induce other terms as well; nevertheless one may hope that for $M \to \infty$, the RG flow will only change the eigenvalues of the projector, but not the property that Φ has only 2 different eigenvalues. A slightly different possibility is

$$S = Tr M(\Phi^2 + c_\Lambda^2{}'(Tr(\Phi))^2)^2 \qquad (4.25)$$

with $M \to \infty$, which amounts to the essentially equivalent constraint $\Phi^2 = -c_\Lambda(Tr(\Phi))^2 \mathbb{1}$. This should be better behaved under renormalization since only marginal operators occur.

Due to the presence of SU($2\mathcal{N}$)-breaking terms e.g. from the Yukawa terms, renormalization will induce additional terms, in particular[12]

$$S = -\frac{N}{g} Tr\left(\phi_0 + \frac{i}{2}\right)^2. \qquad (4.26)$$

As shown in [13], this provides an alternative definition of Yang-Mills on the fuzzy sphere: $F = i\phi_0 - \frac{1}{2}$ is the (scalar) field strength, while the constraint $\Phi^2 = -\frac{1}{4}N^2$ describes precisely 2 tangential gauge fields on S_N^2. This holds provided $Tr(\Phi) \sim iN$, which we assume to follow from the double-trace terms in the action. We therefore expect to find the same physics as discussed in the previous sections. We refrain here from discussing the most general action compatible with the SU($\mathcal{N}$) × SU(2) × SU(2)$_R$ symmetry.

Nevertheless, implementing such a constraint in a 4D quantum field theory is far from trivial. For example, taking $M \to \infty$ appears to be a strong coupling limit; on the other hand, the term $(\Phi^2 + b^2)$ actually vanishes in the fuzzy sphere vacuum, and the desired modes consistent with this constraint are not strongly coupled. Note also that this constraint essentially amounts to a certain type of nonlinear sigma model in 4 dimensions, more precisely to projector-valued quantum fields (up to a shift). This provides renewed motivation to study this type of field theory, as well as an embedding in an extended SUSY model generalizing [9].

Assuming such a constraint for Φ, we can attempt to impose a chirality constraint on the fermions.

5. Chirality constraint: the quantum case

As we have seen explicitly in section 3.4, only in a chiral theory we can expect to get exactly massless fermions. We recall the mechanism: Ψ^{12} resp. Ψ^{21} feel a U(1) magnetic flux on S^2 with strength $k = N_1 - N_2$ resp. $-k$ in the type II vacuum. This leads to k would-be zero modes in Ψ^{12} with positive chirality w.r.t. S^2, and k would-be zero modes in Ψ^{21} with negative chirality w.r.t. S^2. Thus in the chiral case only Ψ^{12} (or only Ψ^{21}) is allowed, and there is no way for it to acquire a 4D mass without further symmetry breaking. In a non-chiral case however, they can pair up and acquire a 4D mass.

Imposing a chirality constraint on the quantum level turns out to be difficult, and we are not able to define a *renormalizable* model which is 6D chiral at the quantum level. The reason is that the 6D chirality operator is a dynamical object which contains the scalar

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[12]the precise values of the constants here are not essential

fields ϕ. This must be so, since at very high energies the model is again 4-dimensional, which is the reason for maintaining renormalizability.

Nevertheless, we discuss some strategies to impose a 6D chirality constraint on the fermions. The most promising approach is to use the modified 6D chirality $\tilde{\Gamma}'$ (3.76) as discussed in section 3.3. For this to be well-defined at the quantum level, the constraint $\Phi^2 = c\mathbb{1}$ seems necessary. Then the definition of $\tilde{\Gamma}'$ simplifies replacing $\tilde{\Phi}$ essentially by Φ. Taking into account possible renormalization of c_Λ, the 6D chirality constraint $\tilde{\Gamma}' = \mathbb{1}$ (3.76) becomes

$$(\Phi + c_\Lambda i\gamma_5)\Psi = 0 \tag{5.1}$$

or $(\Phi - c_\Lambda\,\gamma_5 Tr(\Phi))\Psi = 0$.

There are 2 problems with this approach: first, $\Phi^2 \propto \mathbb{1}$ amounts to some kind of nonlinear sigma model in 4 dimensions, which is not under control to our knowledge. Accordingly, it is not clear if (5.1) can be imposed consistently on the quantum level. The second problem is that the top modes of the diagonal blocks apparently become a large multiplet of massless fermions as discussed in section 3.4, which is clearly undesirable.

Additional fermions. One might try to implement a similar constraint by including additional fermions κ, giving the "wrong" chirality of Ψ a large mass. Consider for example the action

$$S_\kappa = Tr\tilde{\kappa}^T(\Phi\gamma_5 + c'_\Lambda Tr(\Phi)\,)\Psi + S_{\text{kin}}(\kappa) + S_Y(\kappa) \tag{5.2}$$

or[13]

$$S_\kappa = Tr\tilde{\kappa}^T(\chi\gamma_5 + 1\,)\Psi + S_{\text{kin}}(\kappa) + S_Y(\kappa). \tag{5.3}$$

Here κ are fermions in the adjoint of $SU(\mathcal{N})$ which transforms under the global $SU(2) \times SU(2)_R$ as

$$\kappa \to U_R^{-1T}\kappa U^{-1} \tag{5.4}$$

Hence $\tilde{\kappa}^T \to U\tilde{\kappa}^T U_R^T$, and all symmetries are preserved, including the discrete symmetry $\Phi \to -\Phi$. This term couples all modes of Ψ with the "wrong" 6D chirality to the "opposite" modes of κ via terms of the form $Tr\,c_n\,\rho_{\pm,(n),\alpha}\,\kappa^{\pm,(n),\alpha}$, giving them a large mass. While the kinetic term of κ is essentially the same as for Ψ, the Yukawa couplings can be chosen differently.

In this situation, those modes of Ψ which satisfy the 6D chirality constraint do not (or only very weakly) couple to their counterparts in κ, and essentially only their (would-be) zero modes survive. The surviving zero modes of Ψ have a fixed 6D chirality, however those of κ have the opposite chirality. This is quite similar to the non-chiral case, with the additional feature that the Yukawa sector of Ψ and κ may be different. Again, some fine-tuning is required to avoid these zero modes of Ψ and κ to pair up and aquire a mass. This leads again to 2 non-interacting or weakly-interacting almost massless sets of fermions with opposite chirality and quantum numbers.

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[13]The main difference between (5.2) and (5.3) is that the latter doesn't affect the highest mode, which is in fact preferable as discussed in section 3.3

In any case, we end up essentially with a non-chiral model, which requires (mild) fine-tuning in order to have approximately-massless fermions due to the would-be zero modes. Those then come in "mirror pairs" Ψ^{12} resp. Ψ^{21}, i.e. an extra left-handed fermion for each right-handed one with the opposite quantum numbers. Such a scenario is known as *mirror fermions*, and has been considered from the phenomenological point of view in [5]. These models may become chiral at low energies, since the mirror fermions with the "wrong" chirality have different Yukawa sectors and are assumed to be heavier (of the order of the electroweak scale [5]) and hence hidden at low energies.

There are other constraints which could be imposed on the fermions, one of which is described below. However they do not appear to give complex chiral low-energy fermions.

Alternative constraints. One may try to impose a "twisted" version of chirality, in terms of

$$\tilde{R}\Psi := \frac{2i}{N} \Psi \, \Phi, \tag{5.5}$$

assuming again $\Phi^2 \sim \mathbb{1}$. Then imposing a twisted chirality $\tilde{\chi}'\tilde{R} = 1$ amounts to

$$(\tilde{\chi}'\tilde{R} - 1)\Psi = 0 \quad \Longleftrightarrow \quad \Phi\Psi = \Psi\Phi \tag{5.6}$$

while

$$(\tilde{\chi}'\tilde{R} + 1)\Psi = 0 \quad \Longleftrightarrow \quad \Phi\Psi = -\Psi\Phi. \tag{5.7}$$

This relates $SU(2)_V$ to $SU(2)_R$, and preserves only the diagonal $SU(2)_D$ of the twisted picture in section 4.1. Such a constraint could be imposed more easily on the quantum level, by adding

$$TrM\,\Psi_\alpha\{\Phi, \Psi_\beta\}\,\varepsilon^{\alpha\beta} \tag{5.8}$$

to the action with large M. However, our (in-exhaustive) analysis of this and similar constraints did not reveal a chiral low-energy theory. Note that (5.8) is preserved under renormalization; imposing e.g. (5.1) or $\tilde{R}\Psi = \Psi$ is much more difficult.

6. Fluxons and prospects for low-energy SSB

We suggest here briefly a possible mechanism for further (low-energy, electroweak) symmetry breaking. This is speculative at this point, nevertheless it is compelling and natural enough to justify further investigation.

Consider a type II vacuum with an additional fluxon present. Then the scalars for the vacuum can be

$$\phi_a = \begin{pmatrix} \alpha_1 X_a^{(N_1)} \otimes \mathbb{1}_{n_1} & 0 & 0 \\ 0 & \alpha_2 X_a^{(N_2)} \otimes \mathbb{1}_{n_2} & D_a \\ 0 & -D_a^\dagger & c_a \end{pmatrix}, \tag{6.1}$$

where $c_a \in i\mathbb{R}$ denotes the position of the fluxon on S^2, and we assume furthermore a nontrivial off-diagonal column D_a. This is the key to further SSB. To establish (or exclude) such a vacuum would require more detailed analysis of the scalar potential $V(\phi_i)$ which has not been attempted. The crucial point is that any $D_a \neq 0$ can be transformed in the

form $D_a^\dagger = (0, \ldots, 0, d_a^*)$, which implies that $\mathrm{SU}(n_2)$ is broken spontaneously to $\mathrm{SU}(n_2 - 1)$. Assuming that $n_1 = 3, n_2 = 2$ this amounts to further breaking $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{U}(1)$, where D_a plays the role of the "electroweak" Higgs. For the fermions we expect zero modes in the off-diagonal blocks,

$$\psi_{\mathrm{eff}} = \begin{pmatrix} 0 & \psi_{32} & \psi_{31} \\ \psi_{23} & 0 & \psi_{21} \\ \psi_{13} & \psi_{12} & 0 \end{pmatrix} \tag{6.2}$$

while the fluxon sectors are localized on S^2 and therefore essentially 4-dimensional. In particular, $\begin{pmatrix} \psi_{31} \\ \psi_{21} \end{pmatrix}$ corresponds to a fundamental $(5) \to (3) \oplus (2)$ of $\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2)$, providing right-handed quarks and left-handed leptons. ψ_{32} is in the bifundamental of $\mathrm{SU}(3) \times \mathrm{SU}(2)$ corresponding to left-handed quarks. The right-handed leptons might arise from diagonal fluxon block, but this is purely speculative at this point. The family number should arise from the index k, which however appears to prefer $k = 1$. In any case at present this is just a toy model, with the aim to establish the basic mechanisms.

Relation with CSDR scheme. It is interesting to look at the results of this paper from the point of view of coset space dimensional reduction (CSDR). In [3], similar effective 4-dimensional models are constructed starting from gauge theory on $M^4 \times S_N^2$, by imposing CSDR constraints following the general ideas of [27, 28]. These constraints boil down to choosing embeddings of $\mathrm{SU}(2) \subset \mathrm{SU}(\mathcal{N})$, which can be identified with the possible block configurations (2.14). The solutions of the constraints can be formally identified with the lowest modes of the KK-towers of the fields. On the other hand, the present approach takes into account the most general renormalized potential, leading to a vacuum selection mechanism and nontrivial fluxes.

In the context of ordinary CSDR, the question of chiral fermions has been studied in [29, 30]. This appears to be consistent with our conclusion that the model in the present setting is non-chiral. Nevertheless, it is not entirely clear to which extent these commutative results are applicable to the fuzzy case.

7. Discussion

We have presented a simple, renormalizable 4-dimensional $\mathrm{SU}(\mathcal{N})$ gauge theory with suitable scalar and fermionic matter content, which spontaneously develops an extra-dimensional fuzzy sphere. The underlying mechanism is simply SSB and the Higgs effect. The model behaves as a 6-dimensional Yang-Mills theory on $M^4 \times S_N^2$, for energies below a cutoff $\Lambda_{6D} = \frac{N^2}{R}$. The expected KK modes for the fermions are found, extending the bosonic analysis given in [2]. This model is remarkable not only for this striking behavior, but also for a natural mechanism for obtaining an unbroken gauge group $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1)$ as well as zero modes due to a magnetic flux on S_N^2. It represents a particularly simple yet rich realization of the idea of deconstructing dimensions [1], taking advantage of results from noncommutative field theory. This allows to consider ideas

of compactification and dimensional reduction within a renormalizable framework. Our framework provides in fact a slight generalization of the conventional geometric compactification, which manifests itself e.g. in the different effective radii seen by fermions and gauge fields. Moreover, using the results of [4] this mechanism can be understood as an effect of gravity in extra dimensions.

However, it turns out that the model is non-chiral a priori, and imposing a chirality constraint appears to be very difficult on the quantum level. This means that each would-be zero mode from Ψ^{12} has a mirror partner from Ψ^{21}, with opposite chirality and gauge quantum numbers. Thus we arrive essentially at a picture of mirror fermions discussed e.g. in [5] from a phenomenological point of view. While this may still be interesting physically since the "mirror fermions" may have larger mass as the ones we see at low energies, it would be desirable to find a chiral version with similar features. There are indeed many possible directions for generalizations, exploring other types of fuzzy internal spaces. We hope to report on such an extension to $\mathbb{C}P^n_N$ soon.

Another interesting generalization would be supersymmetry. This is natural since both bosons and fermions are in the adjoint of the gauge group; furthermore, the observation of section 4 that both the fermions and scalars are naturally arranged as adjoint of $SU(2\mathcal{N})$ points to supersymmetry at some higher scale. Indeed, a very similar SUSY model has already been discussed in [9] which also develops an extra-dimensional sphere. That model is related to the Maldacena- Nuñez twisted compactification. However, our mechanism for vacuum selection and obtaining a type II vacuum unbroken gauge group $SU(n_1) \times SU(n_2) \times U(1)$ no longer applies in that model, since SUSY is unbroken. This suggests to search for a SUSY version of our model, where the nontrivial type II vacua are accompanied by spontaneous SUSY breaking.

Finally, a natural generalization of this idea is to spontaneously generate not only the extra dimensions but also the "visible" ones. This leads to the matrix-model approach to noncommutative gauge theory, which at least in the Euclidean case has been elaborated in several models such as [20], or e.g. [21] for matrix models related to string theory. Taking into account results in [22], a combination of such models with fuzzy extra dimensions might be particularly interesting.

Acknowledgments

8. The fuzzy sphere

The fuzzy sphere [23] is a matrix approximation of the usual sphere S^2. The algebra of functions on S^2 (which is spanned by the spherical harmonics) is truncated at a given

frequency and thus becomes finite dimensional. The algebra then becomes that of $N \times N$ matrices. More precisely, the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbb{R}^3$ modulo the relation $\sum_{a=1}^3 x_a x_a = r^2$. The fuzzy sphere S_N^2 is the non-commutative manifold whose coordinate functions

$$x_a = r \frac{i}{\sqrt{C_2(N)}} X_a, \qquad x_a^\dagger = x_a \tag{8.1}$$

are $N \times N$ hermitian matrices proportional to the generators of the N-dimensional representation of SU(2). They satisfy the condition $\sum_{a=1}^3 x_a x_a = r^2$ and the commutation relations

$$[X_a, X_b] = \varepsilon_{abc} X_c \ . \tag{8.2}$$

For $N \to \infty$, one recovers the usual commutative sphere. The best way to see this is to decompose the space of functions on S_N^2 into irreps under the SU(2) rotations,

$$\begin{aligned} S_N^2 \cong (N) \otimes (N) &= (1) \oplus (3) \oplus \ldots \oplus (2N-1) \\ &= \{Y^{0,0}\} \oplus \ldots \oplus \{Y^{(N-1),m}\}. \end{aligned} \tag{8.3}$$

This provides at the same time the definition of the fuzzy spherical harmonics Y^{lm}, which we normalize as

$$Tr_N \left((Y^{lm})^\dagger Y^{l'm'} \right) = \delta^{ll'} \delta^{mm'}. \tag{8.4}$$

Furthermore, there is a natural SU(2) covariant differential calculus on the fuzzy sphere. This calculus is three-dimensional, and the derivations of a function f along X_a are given by $e_a(f) = [X_a, f]$. These are essentially the angular momentum operators

$$J_a f = i e_a f = [i X_a, f], \tag{8.5}$$

which satisfy the SU(2) Lie algebra relation

$$[J_a, J_b] = i\varepsilon_{abc} J_c. \tag{8.6}$$

In the $N \to \infty$ limit the derivations e_a become $e_a = \varepsilon_{abc} x_b \partial_c$, and only in this commutative limit the tangent space becomes two-dimensional. For further developments see e.g. [26, 24, 25] and references therein.

9. The spectrum of $\not{D}_{(2)}$

Let us work out the spectrum of $\not{D}_{(2)}$ in detail. For ϕ_a given by (2.15), we have

$$\begin{aligned} \not{D}_{(2)}\Psi &= i\sigma_a(\phi_a\Psi - \Psi\phi_a) + \Psi \\ &= (\alpha\sigma_a J_a + 1)\Psi = \alpha(C_2 - \frac{3}{4} - J^2)\Psi + \Psi, \end{aligned} \tag{9.1}$$

where $C_2 := (\frac{1}{2}\sigma_a + J_a)^2$ is the quadratic Casimir. Ignoring the extra $\mathfrak{su}(n)$ degrees of freedom, this can be evaluated on the decomposition (3.32) using some SU(2) algebra. The eigenvalues of $\not{D}_{(2)}$ on the modes $\Psi_{\pm,(n)}$ are given by

$$\not{D}_{(2)}\Psi_{\pm,(n)} = \left(\alpha\left(C_2 - \frac{3}{4} - J^2 \right) + 1 \right) \Psi_{\pm,(n)}, \tag{9.2}$$

where $C_2 = \frac{1}{4}(n^2 - 1)$, $J^2 = \frac{1}{4}((n \mp 1)^2 - 1)$, and thus

$$\begin{aligned}
\not{D}_{(2)}\Psi_{\pm,(n)} &= \left(\frac{\alpha}{4}(n^2 - (n \mp 1)^2 - 3) + 1\right)\Psi_{\pm,(n)} \\
&= \left(\pm\frac{\alpha}{2}n + (1 - \alpha)\right)\Psi_{\pm,(n)} = E_{\delta=\pm,(n)}\,\Psi_{\pm,(n)}
\end{aligned} \tag{9.3}$$

with

$$E_{\delta=\pm,(n)} \approx \frac{\alpha}{2}\begin{cases} n, \ \delta = 1, & n = 2, 4, \dots, 2N \\ -n, \ \delta = -1, & n = 2, 4, \dots, 2N - 2 \end{cases} \tag{9.4}$$

assuming $\alpha \approx 1$; this is exact for $\alpha = 1$. This can easily be generalized to the type II vacuum, which is not needed however.

Th eigenvalue of χ can be worked out similarly using

$$\chi(\Psi^{\pm}) = \frac{1}{N}\sigma_a(i\phi_a^L + i\phi_a^R) = \frac{1}{N}\left(\alpha_1 J_{1L}^2 - \alpha_2 J_{1R}^2 - \alpha_1^2 X_L^2 + \alpha_2^2 X_R^2\right)\Psi_{\pm,n}, \tag{9.5}$$

where $C_2 = \frac{1}{4}(n^2 - 1)$, $J_{1L,a} = \frac{1}{2}\sigma_a + X_a^L$, and $J_{1R,a} = \frac{1}{2}\sigma_a - X_a^R$. This is written for the case of the type II vacuum. In particular, for the highest mode of the diagonal blocks we have $J_{1L}^2 = \frac{1}{4}((N+1)^2 - 1) = J_{1R}^2$, hence $\chi(\Psi_{+,(2N)}^{12}) = 0$ exactly.

A quick way to determine the chirality for the lowest modes is indicated in the main text.

2D chirality of the conjugate spinors.   Recall from (3.50) that $\overline{\rho}_i^{\dot\alpha} = \varepsilon_{ij}\,\varepsilon^{\dot\alpha\dot\beta}\,(\rho_{\beta,i})^\dagger$, hence $\overline{\rho}^{\dot\alpha} = (i\sigma_2)\,\varepsilon^{\dot\alpha\dot\beta}\,(\rho_\beta)^{\dagger T_2}$ where T_2 denotes transposition of the matrix indices i, j. Then consider

$$\begin{aligned}
\overline{\chi\rho}^{\dot\alpha} &= \varepsilon^{\dot\alpha\dot\beta}(i\sigma_2)(\chi\rho_\beta)^{\dagger T_2} = -\frac{i}{N}\varepsilon^{\dot\alpha\dot\beta}(i\sigma_2)(\sigma_a\{\Phi_a, \rho_\beta\})^{\dagger T_2} \\
&= \frac{i}{N}\varepsilon^{\dot\alpha\dot\beta}(i\sigma_2)\sigma_a^{\dagger T_2}\{\Phi_a, \rho_\beta^{\dagger T_2}\} \\
&= \frac{i}{N}\varepsilon^{\dot\alpha\dot\beta}(i\sigma_2)(-\sigma_2\sigma_a\sigma_2)\left(\{\Phi_a, \rho_\beta^{\dagger T_2}\}\right) \\
&= -\frac{i}{N}\sigma_a\varepsilon^{\dot\alpha\dot\beta}(i\sigma_2)\left(\{\Phi_a, \rho_\beta^{\dagger T_2}\}\right) \\
&= -\chi(\overline{\rho}^{\dot\alpha}),
\end{aligned} \tag{9.6}$$

where we used $\sigma_a^T = -\sigma_2\sigma_a\sigma_2$ and antihermiticity of ϕ_a.

For the zero modes, this can be understood by noting that $\rho \in (2) \otimes (N + m) \otimes (N)$ while $\overline{\rho} \in (2) \otimes (N) \otimes (N + m)$, since the transposition T_2 acts only on the spinor indices.

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